Cohomological obstructions to local-global principle.

§0. Motivation

0.1 (0.2) $\begin{cases} f_1(X_1, \cdots X_n) = 0 \\ \vdots \\ f_m(X_1, \cdots X_n) = 0 \end{cases}$ $f_i(X_1 \cdots X_n) \in k[X_1, \cdots X_n]$ has solution in k?

0.3. The language of algebraic geometry

$$X = \operatorname{Spec}\left(k[X_1, \ldots, X_n] \big/ \langle f_1, \ldots, f_m \rangle \right)$$

$$\{\text{the solution of } (\underline{1.2}) \text{ on } k\} \xleftrightarrow{1:1} X(k)$$

$$X(k) = \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} k, X) = \operatorname{Hom}_{k\text{-alg}}\left(k[X_1, \ldots, X_n] \big/ \langle f_1, \ldots, f_n \rangle, k \right)$$

§1 Rational points on varieties

$X(k)$

1.1 Thm (Model-Weil) A — abelian variety over a global field k. Then $A(k)$ is f.g.

1.2 Thm (Falting) X sm. projective geo. int curve of $g > 1$ over # field k.
Then $\# X(k) < \infty$

1.3 (Weil conjectures)

$X/\mathbb{F}_q$ r.f. scheme. $\overline{X} := X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$

$$Z_X(T) := \exp\left(\sum_{n \geq 1} \# X(\mathbb{F}_{q^n}) \frac{T^n}{n} \right)$$

$$\in \mathbb{Q}[[T]]$$

- (Rationality) $Z_X(T) \in \mathbb{Q}(T)$ as the form $\dfrac{(1-\beta_1 T) \cdots (1-\beta_s T)}{(1-\alpha_1 T) \cdots (1-\alpha_r T)}$ where $\alpha_i, \beta_i \in \overline{\mathbb{Q}}$

• (Riemann hyp.) X sm proper of dim d

$$Z_X(T) = \frac{P_1(T)\,P_3(T)\cdots P_{2d-1}(T)}{P_0(T)\,P_2(T)\cdots P_{2d}(T)}$$

where $P_i(T) \in 1 + T\mathbb{Z}[T]$

factors over $\mathbb{C}$ as $\prod_{j=1}^{b_i} (1-\underline{\alpha_{ij}}\,T)$

and for all archimedian on $\mathbb{Q}(\alpha_{ij})$

one has $|\alpha_{ij}| = q^{\frac{i}{2}}$

$b_i := \deg P_i$ —

$$P_i(T) = \det(1 - FT, H^i(\bar{X}, \mathbb{Q}_l))$$

$b_i := \dim_{\mathbb{Q}_l}(H^i(\bar{X}, \mathbb{Q}_l))$

independent of $l \neq p$

$$\# X(\mathbb{F}_{q^n}) = \sum_i (-1)^i \operatorname{Tr}(F^n, H^i(\bar{X}, \mathbb{Q}_l))$$

- (fcn. eq.) $\quad Z_X\left(\frac{1}{q^d T}\right) = \pm q^{\frac{d\chi}{2}} T^{\chi} Z_X(T)$

where $\chi := b_0 - b_1 + b_2 - \cdots + b_{2d} \in \mathbb{Z}$

Euler char.

$$"\pm" = (-1)^{N^+}$$

where $N^+ = \#\left(\text{eigen}\left(F \curvearrowright H^d(\bar{X}, \mathbb{Q}_\ell)\right) = q^{\frac{d}{2}}\right)$

- (<u>Betti #</u>) $b_i := \dim_{\mathbb{Q}_\ell}\left(H^i(\bar{X}, \mathbb{Q}_\ell)\right)$

if $\exists\ \mathfrak{X}$ sm. proper f.t. $\mathbb{Z}_q$-schem

with $X \cong \mathfrak{X} \times_{\mathbb{Z}_q} \mathbb{F}_q$

fixing an embedding $\mathbb{Z}_q \subset \bar{\mathbb{Z}}_q \subset \bar{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$

Deligne 1973 $\quad X_{\mathbb{C}} := \mathfrak{X} \times_{\mathbb{Z}_q} \mathbb{C}$ then

$$H^i_{\mathrm{Betti}}(X_{\mathbb{C}}(\mathbb{C}), \mathbb{Z}_l) \xrightarrow{\sim} H^i(\bar{X}, \mathbb{Z}_l)$$

$$\parallel$$

$$H^i(X_{\mathbb{C}}(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_l$$

$$\left(H^i(\bar{X}, \mathbb{Z}_l) = \varprojlim_n H^i_{\text{ét}}(\bar{X}, \mathbb{Z}/l^n\mathbb{Z}) \right)$$

$$H^i(\bar{X}, \mathbb{Q}_l) \simeq H^i(\bar{X}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

In particular, $b_i = \operatorname{rank} H^i(X_{\mathbb{C}}(\mathbb{C}), \mathbb{Z})$

1.4. Local-global principle

$k = \mathbb{Q}$. $\quad X(\mathbb{Q})$. $\quad X$ projective.

$$\underline{X(\mathbb{Z})} \neq \emptyset \Rightarrow X(\mathbb{Z}/p\mathbb{Z}) \neq \emptyset, \ \forall p \leq \infty$$

$$X(\mathbb{R}) \neq \emptyset.$$

$$\mathbb{Z}/p^2\mathbb{Z} \cdots \rightsquigarrow \mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$$

$$X(\mathbb{Z}) \subseteq \underline{X(\mathbb{R}) \times \prod_{p<\infty} X(\mathbb{Z}_p)}$$

k — field.

$\underline{k} \subset \prod_v' k_v = \mathbb{A}_k$ $\underline{O_v \subset k_v}$

1.5 Def. (i). Let k be a field. a variety over k is a seperated k-scheme of f.t.
$(X(O_v) \hookrightarrow X(k_v))$

(iii) k a global field X/k var

$X(k)$ —— rational point

$X(\mathbb{A}_k)$ —— adèlic point.

16. Rmk. (i) If X is proper (say, proj)

then $X(\mathbb{A}_k) = \prod_v X(k_v)$

local point

(ii). $X(\mathbb{A}_k)$ has a natural topology

$k_v \rightsquigarrow X(k_v) \hookrightarrow X(\mathbb{A}_k)$

$X(k) \hookrightarrow X(\mathbb{A}_k)$ approximations.

For (non-proper) X,

$$X(\mathbb{A}_k) = \mathrm{Hom}_{\mathrm{Spec}\,k}(\mathrm{Spec}\,\mathbb{A}_k, X)$$

$$\cong$$

$$\prod_v{}' X(k_v) \quad \left(\text{w.r.t. } \mathcal{X}_v(\mathcal{O}_v)\right)$$

integral model $(\mathcal{X}_v \times_{\mathcal{O}_k} k_v = X_{k_v})$

$X_{\mathbb{A}_k}$ $\quad$ $\mathrm{Spec}\,\mathcal{O}_v \to \mathrm{Spec}\,k$

1.7 $X(k) \subseteq X(\mathbb{A}_k)$.

$X(k) \neq \emptyset \Rightarrow X(\mathbb{A}_k) \neq \emptyset$.

what about the converse?

1.8 ~~Def~~. If $X(\mathbb{A}_k) \neq \emptyset \Rightarrow X(k) \neq \emptyset$

(local-global)

we say Hasse-principle holds for X/k

HP

1.9 thm (Hasse-Minkowski) For X/k defined by quadratic forms, HA always holds.

1.10. Counter-example

$$X : \quad 3X^3 + 4Y^3 + 5Z^3 = 0 \quad / \mathbb{Q}.$$

HA fails.

$$\prod_{p \le \infty} X(\mathbb{Q}_p) \neq \emptyset \quad \text{but} \quad X(\mathbb{Q}) = \emptyset$$

$$X(\mathbb{Q}) \subseteq \qquad \subseteq X(\mathbb{A}_k)$$

obstruction ....

§2. Brief review of étale cohomology

2.1 Motivation.

- **Weil conj** $\quad H^i(X_{\mathbb{C}}(\mathbb{C}), \mathbb{Q})$

$\vdots$

topolgy inh from $\mathbb{C}$ ~~top~~

$X/\mathbb{F}_q$ $\quad$ Zar. but too course

$$\rightsquigarrow H_c^i(\bar{X}, \mathbb{Q}_\ell) = \varprojlim H^i_{c,et}(\bar{X}, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

- <u>Gal coh</u>

$$\Gamma = \mathrm{Gal}(k^s/k)$$

$$\Gamma\text{-module} \xrightarrow{\sim} \mathrm{Ab}(k_{\acute{e}t})$$

(abelian) étale sheaves on k

$$M_{\mathcal{F}} = \varinjlim_{K/k \text{ fin sep}} \mathcal{F}(\operatorname{Spec} K) \longleftarrow\!\shortmid \mathcal{F}$$

$$\rightsquigarrow H^i_{\acute{e}t}(k, \mathcal{F}) \cong H^i(\Gamma_k, M_{\mathcal{F}})$$

- Singular $X/\mathbb{C}$

$$H^i(X_{\mathbb{C}}(\mathbb{C}), F) \cong H^i_{\text{ét}}(X, F)$$

finite ab gp

2.2 Grothendieck (pre-)topology

- $\mathcal{C}$ category. A Groth. pre-top T on $\mathcal{C}$ is collection of

$\{ U_i \to U \}_{i \in I}$ in $\mathcal{C}$

↳ called <u>covering</u>

s.t.

- $\{ U' \xrightarrow{\sim} U \} \in T$
- If $\{ U_i \to U \}$, $\{ V_{ij} \to U_i \} \in T$

  then $\{ V_{ij} \to U \} \in T$
- If $\{ U_i \to U \} \in T$: base change

by $V \to U$ then

$$\{ V \times_U U_i \longrightarrow V \} \in T$$

$(\mathcal{C}, T)$ is (Groth.) site.

· Eg – (small étale site on X) $X_{ét}$

X scheme. $\mathcal{C}$ = cat. of étale X-schemes

$$\begin{tikzcd} Y' \arrow[r, "ét"] \arrow[d] & X \\ Y \arrow[ur, "ét"'] & \end{tikzcd}$$

$\{U_i \to U\}$ is a covering if $\coprod_i U_i \to U$ surj. ét

($Y \to X$ is étale if it is flat and unramified

e.g K/k $\operatorname{Spec} K \to \operatorname{Spec} k$

$Y \to X = \operatorname{Spec} k$ is étale (+ f.t.)

iff $Y = \coprod_{\lambda}^{(n)} \operatorname{Spec} K_\lambda$
$(i = 1)$

K_λ / k fin sep ext!)