

cohomological obstructions to
local-global principle. (2)

2.2 Grothendieck top.

- (small Zariski site) X_{zar}

$\mathcal{C} =$ (open immersion $U \hookrightarrow X$)

coverings. $\{U_i \rightarrow U\}; \quad \bigcup_i U_i = U$

- (small étale site) $X_{\text{ét}}$.

$$\mathcal{C} = \left(U \xrightarrow{\text{ét}} X \right)$$

covering $\{U_i \rightarrow U\}_i$:
$$\coprod_i U_i \longrightarrow U$$
 surjective étale

Ex. $X = \text{Spec } k$ $k_{\text{ét}}$
 $U \rightarrow k$ ét. (f.t)

$U = \coprod_i \text{Spec } K_i$ K_i/k fin sep ext.

- (Big fppf (fpqc) site)

X_{fppf} (X_{fpqc}).

$Y \rightarrow X$ is fppf (fpqc) if

it is faithfully flat and locally of

finite presentation (faithfully flat

and "quasi-compact" (every qc open of X

is a image of some qc open of Y))

big $e = \frac{SCh/X}{\text{---}}$

covering $\{U_i \rightarrow U\}_i$ if

$\coprod_i U_i \rightarrow U$ is fppf (fpgc).

— Inclusion yields continuous map of sites

$$\underline{\underline{X_{fpgc}}} \rightarrow X_{fppf} \rightarrow X_{\text{ét}} \rightarrow \underline{\underline{X_{\text{zar}}}}$$

$$\begin{array}{ccc}
 U_1 & \xrightarrow{P_2} & U_0 \\
 P_1 \downarrow & \square & \downarrow \\
 U_0 & \rightarrow & U_{-1}
 \end{array}$$

§ 2.3 Cohomology on sites

[SP] ...

• \mathcal{C} a site. $\text{Sh}(\mathcal{C}) = \text{Fun}(\mathcal{C}^{\text{op}}, \text{Set})$

• For a covering $\{U_i \rightarrow U\}$.

Let $U_{-1} = U$, $U_0 = \coprod_i U_i \rightarrow U_{-1}$

$U_1 = U_0 \times_{U_{-1}} U_0$... $U_n = U_0 \times_{U_{-1}} \dots \times_{U_{-1}} U_0$

$\underbrace{\hspace{10em}}_{n+1}$

$\pi \text{---} \cancel{P(U_i)} \quad P(U_0)$

A presheaf \mathcal{P} : $\mathcal{C}^{op} \rightarrow \text{Set}$.

is a sheaf if

- \mathcal{P} preserves products

$$(i.e. \mathcal{P}(\prod_i U_i) = \prod_i \mathcal{P}(U_i))$$

- (descent condition) for all covering $\{U_i \rightarrow U\}$
 $U_{-1} = U, U_0 = \dots$

we have

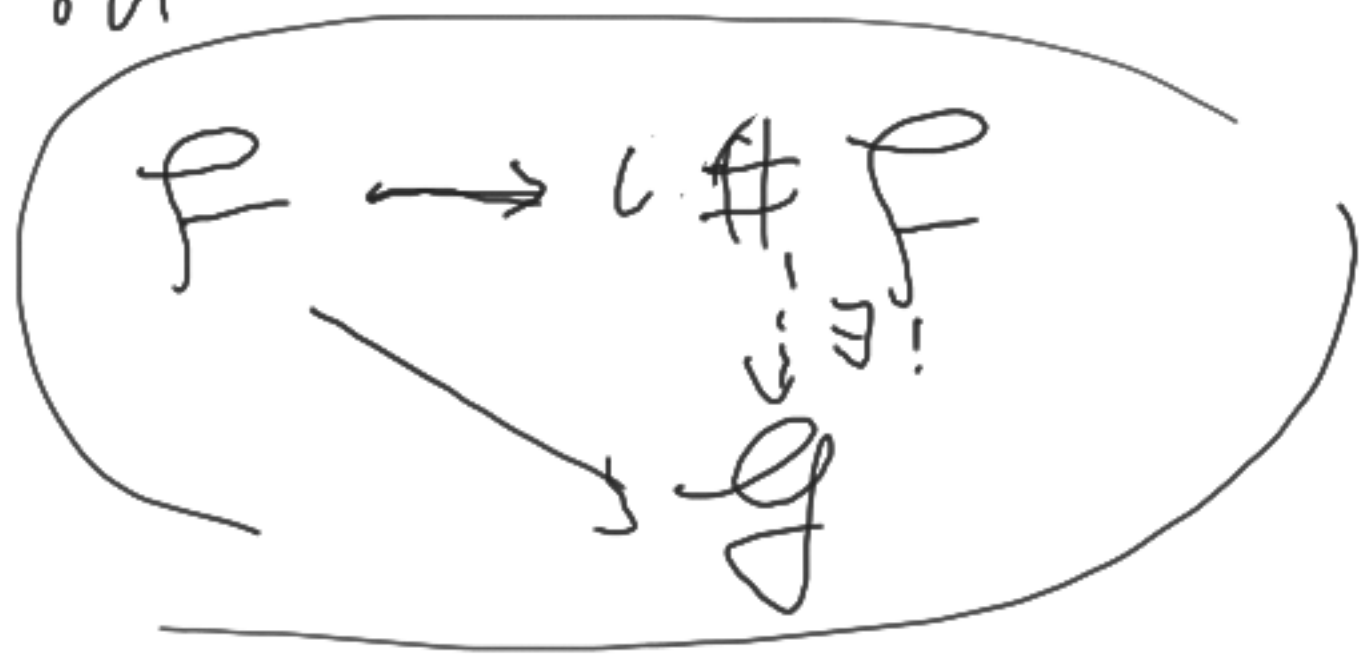
$$\mathcal{P}(U_{-1}) \cong \lim_{\substack{\rightarrow \\ \mathcal{P}^*}} (\mathcal{P}(U_0) \xrightarrow{\mathcal{P}^*} \mathcal{P}(U_1))$$

(or equalizer.)

(explicitly if $P(u_1) \leftrightarrow P(u_2) \not\Rightarrow P(u_1)$)
 call P separated

show $(C) \hookrightarrow \text{PSH}(C)$ admits a
 left adjoint $\# : \text{PSH}(C) \rightarrow \text{Sh}(C)$

$$\therefore \text{Hom}_{\text{PSH}}(F, \mathcal{G}) \cong \text{Hom}_{\text{Sh}}(\#F, \mathcal{G})$$



Explicitly, $\#$ can be constructed as follows:

- For $\mathcal{P} \in \mathcal{PSh}(\mathcal{C})$, let

$$\mathcal{P}^+ : U \longmapsto \text{colim} \left(\lim_{\substack{U_0 \rightarrow U_i = U \\ \text{covering}}} (\mathcal{P}(U_0) \rightrightarrows \mathcal{P}(U_i)) \right)$$

then $\mathcal{P}^+ \in \mathcal{PSh}(\mathcal{C})$

s.t. $\textcircled{1}$ \mathcal{P}^+ is separated
 $\textcircled{2}$, \mathcal{P} is separated $\Rightarrow \overline{\mathcal{P}^+}$ is a sheaf.

$$\rightsquigarrow \mathcal{P}^\# := \mathcal{P}^{++} \quad \checkmark$$

• $\mathcal{P} \in \text{Psh}(\mathcal{C})$. $\Gamma(\mathcal{C}, -) : \text{Psh}(\mathcal{C})$

the global section $\Gamma(\mathcal{C}, \mathcal{P})$ ↓ Set

$:= \text{Mor}_{\text{Psh}(\mathcal{C})}(*, \mathcal{P})$

$\mathcal{P} : \mathcal{U} \rightarrow *$
 $* \in \text{Psh}(\mathcal{C})$
~~final~~ obj.

Push if $X \in \mathcal{C}$ is final \Rightarrow

$\Gamma(\mathcal{C}, \mathcal{P}) = \Gamma(X, \mathcal{P})$
 $:= \mathcal{P}(X)$ ✓

$\mathcal{C} = X \text{ set}$ X final

• $Ab(\mathcal{C}) \subseteq \underline{Shw}(\mathcal{C}) \subseteq \underline{Psh}(\mathcal{C})$

" \rightarrow is a abelian cat.

$F: \mathcal{C}^{op} \rightarrow Ab$

$\Gamma(\mathcal{C}, -): Ab(\mathcal{C}) \rightarrow Ab$

||s

How $Ab(\mathcal{C}) (\mathbb{Z}, -)$ left exact.

\rightsquigarrow Right derived functor $\underline{D}^+(Ab(\mathcal{C}))$

$R\Gamma(\mathcal{C}, -): \underline{D}^+(\mathcal{C}) \rightarrow \underline{D}^+(Ab)$

$(\text{as } \mathcal{F}^n \rightarrow \mathcal{F}^{n+1} \rightarrow \dots \quad \mathcal{F}^i \in Ab(\mathcal{C}))$

$X \in D^+(\mathcal{C})$, choose a

injective resolution $I \in D^+(\mathcal{C})$

$$X \xrightarrow{q_i's} I \quad R\Gamma(\mathcal{C}, -) = \Gamma(\mathcal{C}, I)$$

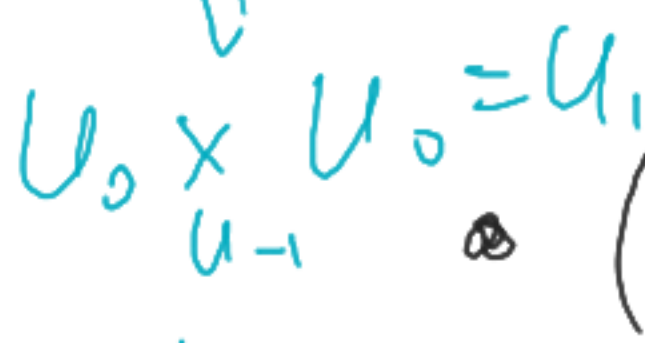
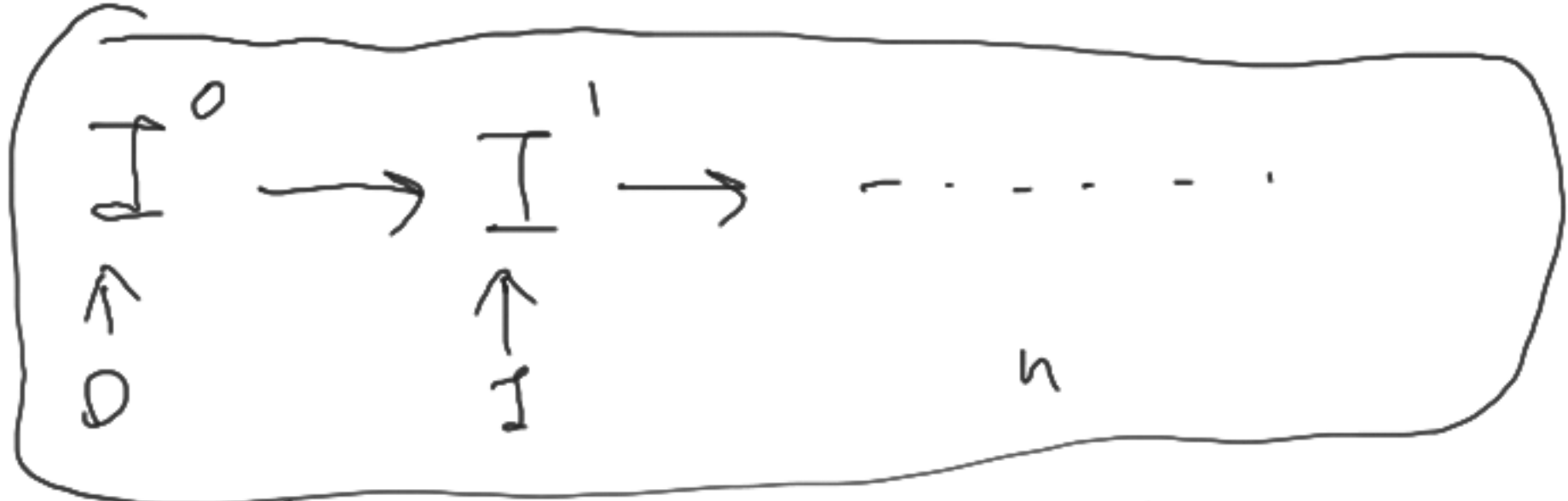
$$H^i(\mathcal{C}, -) := H^i R\Gamma(\mathcal{C}, -) : \text{Ab}(\mathcal{C})$$

eg

$$\begin{array}{ccc} H^i_{\tau}(X, \mathbb{F}) & \xrightarrow{\quad} & \tau = \bar{e}_1 / \text{fppf} \\ \parallel & \searrow & \downarrow \\ H^i(X_{\tau}, \mathbb{F}) & & \text{Ab} \end{array}$$



$$H_{\text{ét}}^n(X, \mathcal{F}) = H^n(\mathcal{F}(X, \mathcal{I}^\bullet))$$



(Čech cover) $U_0 \rightarrow U_{-1}$ covering



Čech complex.

$$P \in \text{AbPsh}(\mathcal{C})$$

$$\mathcal{C}^\bullet \quad 0 \rightarrow P(U_0) \rightrightarrows P(U_{-1}) \rightrightarrows P(U_2) \dots$$

the differential map is obtained by alternating sum of

$$\check{H}^i(U_0/U_{-1}, \mathcal{P}) := H^i(\mathcal{C}^\bullet)$$

$$\check{H}^i(U, \mathcal{P}) := \text{colim}_{U_0 \rightarrow U_{-1}} \check{H}^i(U_0/U_{-1}, \mathcal{P})$$

$U_0 \rightarrow U_{-1}$
 covering U

§3 : Cohomological obstructions

3.1 Brauer groups and rational pts

• Recall that for a field k

$$B_r k = H^2(k, \overline{k}^X)$$

eg (CFT)

$$B_r \mathbb{C} = 0$$

$$B_r \mathbb{R} = \frac{1}{2} \mathbb{R} / \mathbb{Z}$$

$$B_r k \simeq \frac{1}{\text{inv}} \oplus \mathbb{Z} / \mathbb{Z}$$

• For any scheme X , define

$$\text{Br } X = H_{\text{ét}}^2(X, \mathbb{Q}_{m, X})$$

\rightsquigarrow $\text{Pw} : \text{Sch}^{\text{op}} \rightarrow \text{Ab}$ a functor

$$Y \rightarrow X \rightsquigarrow H_{\text{ét}}^2(X, \mathbb{Q}_{m, X})$$

$$\downarrow$$

$$H_{\text{ét}}^2(Y, \mathbb{Q}_{m, Y})$$

$$\mathbb{Q}_{m, X} := \text{Spec } \mathbb{Z}[T, T^{-1}] \times_{\text{Spec } \mathbb{Z}} X$$

$$\text{Qm}, X : \text{Sch}_X^{\text{op}} \rightarrow \text{Ab}$$

$$U \longmapsto$$

$$\text{Hom}_X(U, \mathbb{Q}_{m, X}) = \mathcal{O}_U^X(U)$$

$$U/k, \underline{\quad} \quad k[U]^{X}$$

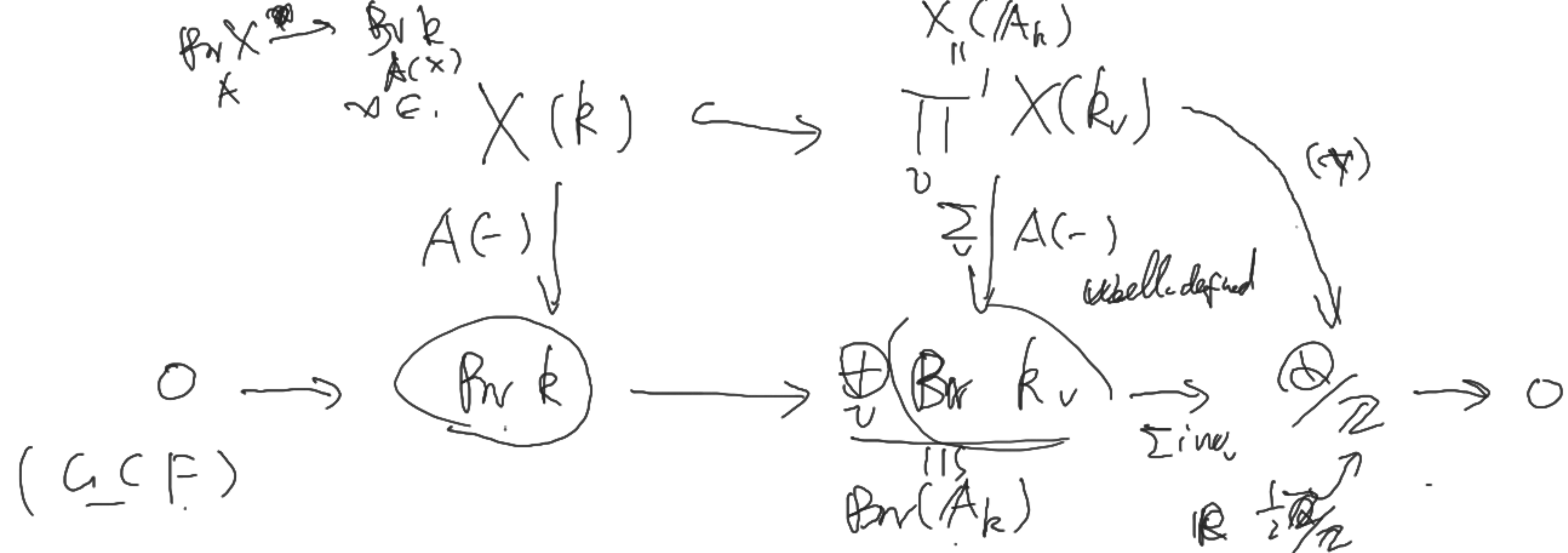
$$h_X = X : \text{Schemes} \rightarrow \text{Set}$$

$$h_X \in \text{shv}(\text{S-fppf})$$

$$\circlearrowleft \quad X/k \quad x \in X(k) \quad \rightsquigarrow \quad x^* : \text{pt} X \rightarrow \text{pt} k$$

$$\text{Spec } k \xrightarrow{x} X \quad A \mapsto A(x)$$

\circlearrowleft k # field, X/k var. $A \in \text{pt} X$



Let $A \in \text{Pr } X$ variables, $(*) \rightsquigarrow$ pairing

$\langle -, - \rangle_{\text{EM}} : \frac{\text{Pr } X}{A} \times \frac{X(A_k)}{(x_v)} \longrightarrow \bigoplus \mathbb{Z}$
 $\longmapsto \sum_v \nu_v A(x_v)$

Defn $X(A_k)^{\text{Br}} = \left\{ (x_i) \in X(A_k) \mid \right.$

$\left. \langle A, \mathbb{Z} \rangle_{\text{Br}} = 0, \forall A \in \text{Br} X \right\}$

$X(k)$

$$X(k) \subseteq \underline{X(A_k)^{\text{Br}}} \subseteq X(A_k)$$

Brauer-Manin obstruction

G in comm gp scheme

$$\text{Br } G = H^2(G, \mathbb{Q}/\mathbb{Z})$$

$$\text{for } G : \begin{matrix} \times \\ \hline \end{matrix} C_1(A_k) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

g_1, g_2

$$A_k \xrightarrow{g_1, g_2} C_1 \rightsquigarrow \text{Br}(G) \longrightarrow \text{Br } A_k$$

