cohomological obstructions to
local-global principle (2)

2.2 Grothendieck top.

- (small Zariski site) X_{Zar}

$\mathcal{C} = $ (open inversion $U \hookrightarrow X$)

coverings. $\{U_i \to U\}_i$ $\quad \bigcup_i U_i = U$

- (small étale site) $X_{\text{ét}}$.

$$\mathcal{C} = (\quad U \xrightarrow{\text{ét}} X \quad)$$

covering $\{U_i \to U\}_i$: $\coprod_i U_i \longrightarrow U$ surjective étale

Eg. $X = \operatorname{Spec} k$ $\quad k_{\text{ét}}$

$U \longrightarrow k$ ét. (f.t)

$U = \coprod_{i \text{ (finite)}} \operatorname{Spec} K_i$ $\quad K_i/k$ fin sep ext.

– (Big fppf (fpqc) site)

X_{fppf} (X_{fpqc})

$Y \to X$ is fppf (fpqc) if

it is faithfully flat and locally of

finite presentation (faithfully flat

and "quasi-compact" (any qc open of X

is a image of some qc open of Y))

big $\mathcal{C} = Sch/X$

covering $\{U_i \to U\}_i$ if

$\coprod_i U_i \longrightarrow U$ is fppf (fpqc)

– Inclusion yields contious map of sites

$$X_{fpqc} \longrightarrow X_{fppf} \longrightarrow X_{ét} \longrightarrow X_{zar}$$

$$\begin{tikzcd} U_1 \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\square"] & U_0 \arrow[d] \\ U_0 \arrow[r] & U_{-1} \end{tikzcd}$$

§ 2.3 Cohomology on sites

[SP] ...

- $\mathcal{C}$ a site. $PSh(\mathcal{C}) = Fun(\mathcal{C}^{op}, Set)$
- For a covering $\{U_i \to U\}$.

Let $U_{-1} = U$, $U_0 = \coprod_i U_i \to U_{-1}$

$U_1 = U_0 \times_{U_{-1}} U_0$ . ..... $U_n = \underbrace{U_0 \times_{U_{-1}} \cdots \times_{U_{-1}} U_0}_{n+1}$

$\prod \cancel{P(U_i)}$ $P(U_0)$

A presheaf $\mathcal{P} : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$.

is a <u>sheaf</u> if

— $\mathcal{P}$ preserves products

(i.e. $\underline{\mathcal{P}(\coprod_i U_i)} = \prod_i \mathcal{P}(U_i)$)

— (descent condition) for all covering $\{U_i \to U\}_i$

$U_{-1} = U$, $U_0 = \cdots$

we have

$$\mathcal{P}(U_{-1}) \cong \lim \left(\mathcal{P}(U_0) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathcal{P}(U_1) \right)$$

(or equalizer.)

(explicitly call $\mathcal{P}$ separated if $\mathcal{P}(U_{-1}) \hookrightarrow \mathcal{P}(U_0) \cancel{\rightrightarrows \mathcal{P}(U_1)}$)

$\mathrm{Shv}(\mathcal{C}) \xhookrightarrow{\iota} \mathrm{PSh}(\mathcal{C})$ admits a left adjoint $\# : \mathrm{PSh}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{C})$

i.e. $\mathrm{Hom}_{\mathrm{PSh}}(\mathcal{F}, \iota\mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{Sh}}(\#\mathcal{F}, \mathcal{G})$

$$\begin{tikzcd}
\mathcal{F} \arrow[r, hook] \arrow[dr] & \iota\#\mathcal{F} \arrow[d, dashed, "\exists!"] \\
& \iota\mathcal{G}
\end{tikzcd}$$

Explicitly, $\#$ can be constructed as follows:

- For $P \in PSh(\mathcal{C})$, let

$$P^+ : U \longmapsto \operatorname*{colim}_{\substack{U_0 \to U_{-1} = U \\ \text{covering}}} \left(\lim \left(P(U_0) \rightrightarrows P(U_1) \right) \right)$$

then $P^+ \in PSh(\mathcal{C})$

s.t. ① P^+ is seperated

② P is seperated $\Rightarrow$ P^+ is a sheaf.

$\rightsquigarrow$ $P^\# := P^{++}$ ✓

- $P \in Psh(\mathcal{C})$. $\Gamma(\mathcal{C}, -) : Psh(\mathcal{C}) \downarrow Set$

the global section $\underline{\Gamma(\mathcal{C}, P)}$.

$:= Mor_{Psh(\mathcal{C})}(*, P)$

$U \in \mathcal{C}$

$P : U \longmapsto *$

$* \in Psh(\mathcal{C})$

final obj.

Rmk if $X \in \mathcal{C}$ is final $\Rightarrow$

$\Gamma(\mathcal{C}, P) = \Gamma(X, P)$
$:= P(X)$ ✓

$\mathcal{C} = X_{\acute{e}t}$. $\underline{X \text{ final}}$

• $\mathrm{Ab}(\mathcal{C}) \subseteq \underline{\mathrm{Sh}}(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ is a abelian cat.

$\quad \shortparallel$

$\mathcal{F} : \mathcal{C}^{op} \to \mathrm{Ab}$

$$\Gamma(\mathcal{C}, -) : \mathrm{Ab}(\mathcal{C}) \longrightarrow \mathrm{Ab}$$

$$\cong$$

$$\mathrm{Hom}_{\mathrm{Ab}(\mathcal{C})}(\mathbb{Z}, -) \quad \text{Left exact.}$$

$\rightsquigarrow$ Right derived functor

$$R\Gamma(\mathcal{C}, -) : \underbrace{D^+(\mathcal{C})}_{\shortparallel} \overset{D^+(\mathrm{Ab}(\mathcal{C}))}{} \longrightarrow D^+(\mathrm{Ab})$$

$$(\mathcal{F}^n \to \mathcal{F}^{n+1} \to \cdots\cdots \quad \mathcal{F}^i \in \mathrm{Ab}(\mathcal{C}))$$

| $X \in D^+(\mathcal{C})$, choose a

injective resolution $I \in D^+(\mathcal{C})$

$X \xrightarrow{\text{qis}} I$. $\quad R\Gamma(\mathcal{C}, -) = \Gamma(\mathcal{C}, I)$

$H^i(\mathcal{C}, -) := H^i R\Gamma(\mathcal{C}, -) : Ab(\mathcal{C}) \to Ab$.

Eg $H^i_\tau(X, F) = H^i(X_\tau, F)$, $\tau = $ ét / fppf.

$$0 \to \mathcal{F} \to \boxed{\begin{matrix} I^0 & \to & I^1 & \to & \cdots \\ \uparrow & & \uparrow & & \\ 0 & & 1 & & n \end{matrix}}$$

$$H^n_{sf}(X, \mathcal{F}) = H^n\big(\Gamma(X, I^\bullet)\big)$$

$U_0 \times_{U_{-1}} U_0 \times_{U_{-1}} U_0 = U_2$

$U_0 \quad U_0 \quad U_0$

$U_0 \times_{U_{-1}} U_0 = U_1$ $\circledast$ (Čech coh) $U_0 \to U_{-1}$ covering

$U_0 \rightsquigarrow$ Čech complex. $P \in \mathrm{AbPSh}(\mathcal{C})$

$\mathcal{C}^\bullet \qquad 0 \to P(U_0) \rightrightarrows P(U_1) \overset{\to}{\rightrightarrows} P(U_2) \cdots$

the differential map is obtained by alternating sum of

$$\check{H}^i\left(\mathcal{U}_0 / \mathcal{U}_{-1}, \mathcal{P} \right) := H^i(\mathcal{C}^\bullet)$$

$$\check{H}^i(\mathcal{U}, \mathcal{P}) := \operatorname*{colim}_{\substack{\mathcal{U}_0 \to \mathcal{U}_{-1} \\ \text{covering} \quad \| \\ \mathcal{U}}} \check{H}^i\left(\mathcal{U}_0 / \mathcal{U}_{-1}, \mathcal{P} \right)$$

§3 : Cohomological obstructions

3.1. Brauer gps and rational pts

- Recall that for a field k

$$\mathrm{Br}\, k = H^2(k, \bar{k}^\times)$$

<u>Eg</u> (CFT) $\quad \overline{\mathrm{Br}\, \mathbb{C} = 0} \quad \mathbb{G}_m$

$\mathrm{Br}\, \mathbb{R} = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$

$$\mathrm{Br}\, k_v \xrightarrow[\mathrm{inv}_v]{\sim} \mathbb{Q}/\mathbb{Z}$$

- For any scheme X, define

$$\mathrm{Br}\, X = H^2_{ét}(X, \mathbb{G}_{m,X})$$

$$\rightsquigarrow \quad \mathrm{Br} : \mathrm{Sch}^{op} \longrightarrow \mathrm{Ab} \quad \text{a functor}$$

$$Y \to X \quad \rightsquigarrow \quad \begin{array}{c} H^2_{ét}(X, \mathbb{G}_{m,X}) \\ \downarrow \\ H^2_{ét}(Y, \mathbb{G}_{m,Y}) \end{array}$$

$$\mathbb{G}_{m,X} := \mathrm{Spec}\, \mathbb{Z}[T, T^{-1}] \times_{\mathrm{Spec}\, \mathbb{Z}} X$$

$$\mathbb{G}_{m,X} : \mathrm{Sch}_X^{op} \longrightarrow \mathrm{Ab}$$

$$U \longmapsto \mathrm{Hom}_X(U, \mathbb{G}_{m,X}) = \mathcal{O}_U^\times(U)$$

$$U/k. \qquad \qquad \qquad \qquad \qquad \qquad \overset{\|}{k[U]^\times}$$

$h_X = X : \mathrm{Sch}_{/S}^{op} \longrightarrow \mathrm{Set}$

$h_X \in \mathrm{shv}(S_{fpqc})$

$\circledast \hookrightarrow$ $X_{/k}$ $x \in X(k)$ $\rightsquigarrow x^* : \mathrm{Pr}\,X \longrightarrow \mathrm{Pr}\,k$

$\mathrm{Spec}\,k \xrightarrow{x} X$ $A \longmapsto A(x)$

$\circledast$ k # field. $X_{/k}$ var. $A \in \mathrm{Pr}\,X$

$\mathrm{Br}\, X \to \mathrm{Br}\, k$, $A \mapsto A(x)$, $x \in$.

$$
\begin{tikzcd}
 & X(k) \arrow[r, hook] \arrow[d, "A(-)"'] & \prod_v' X(k_v) = X(\mathbb{A}_k) \arrow[d, "\sum_v A(-)"', "\text{well-defined}"] \arrow[dr, bend left, "(*)"] & & \\
0 \arrow[r] & \mathrm{Br}\, k \arrow[r] & \bigoplus_v \mathrm{Br}\, k_v \cong \mathrm{Br}(\mathbb{A}_k) \arrow[r, "\sum \mathrm{inv}_v"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}
$$

(GCF)

$\mathbb{R}$ $\tfrac{1}{2}\mathbb{Z}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$

Let $A \in \mathrm{Br}\, X$ varies, $(*) \rightsquigarrow$ pairing

$$\langle -,- \rangle_{BM} : \mathrm{Br}\, X \times X(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}$$

$$(A, (x_v)) \mapsto \sum_v \mathrm{inv}_v A(x_v)$$

Define $X(\mathbb{A}_k)^{\mathrm{Br}} = \left\{ (x_v) \in X(\mathbb{A}_k) \;\middle|\; \langle A, x_v \rangle_{\mathrm{Br}} = 0,\ \forall A \in \mathrm{Br}X \right\}$

$X(k) \subseteq$

$$X(k) \subseteq \underline{X(\mathbb{A}_k)}^{\mathrm{Br}} \subseteq X(\mathbb{A}_k)$$

<u>Brauer–Manin</u> obstruction

G in comm gp scheme

$$\mathrm{Br}\, G = H^2(G, \mathbb{G}_m)$$

$$\mathrm{Br}\, G \times G(\mathbb{A}_k) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

$$g_1 \cdot g_2$$

$$\mathbb{A}_k \xrightarrow{g_1 \cdot g_2} G \rightsquigarrow \quad \mathrm{Br}(G) \longrightarrow \mathrm{Br}\, \mathbb{A}_k$$