Cohomological obstructions to local-global principle.

Review of BM pairing.

$x_v \in X(k_v)$

$$\begin{array}{ccccc} \operatorname{Spec} k_v & \xrightarrow{x_v} & X & & \\ \operatorname{Br} X & \xrightarrow{x_v^*} & \operatorname{Br} k_v & \xrightarrow{\mathrm{inv}_v} & \mathbb{Q}/\mathbb{Z} \\ A & \longmapsto & A(x_v) & \longmapsto & \mathrm{inv}_v A(x_v) \end{array}$$

$$
\begin{array}{ccccc}
\langle -, - \rangle_{BM} : & \mathrm{Br}\,X & \times & X(\mathbb{A}_k) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\
 & A & & (x_v)_v & \longmapsto & \sum_v \mathrm{inv}_v\, A(x_v) \\
 & & & X(k) & \longmapsto & 0
\end{array}
$$

finite sum

$$
0 \longrightarrow \mathrm{Br}\,k \longrightarrow \underbrace{\bigoplus_v \mathrm{Br}\,k_v}_{\cong\, \mathrm{Br}(\mathrm{Spec}\,\mathbb{A}_k)} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0
$$

$$
X(k) \subseteq \boxed{X(\mathbb{A}_k)^{\mathrm{Br}}} = \left\{ (x_v) \in X(\mathbb{A}_k) \;\middle|\; \langle A, (x_v) \rangle = 0,\ \forall A \in \mathrm{Br}\,X \right\}
$$

$$
X(\mathbb{A}_k)^{\mathrm{Br}} \subseteq X(\mathbb{A}_k)
$$

3.2 The F - obstruction

Let $F : (Sch_{/k})^{op} \longrightarrow Set$ be a functor

$T \xrightarrow{x} X$

T — k- scheme. $\forall\ x \in X(T)$

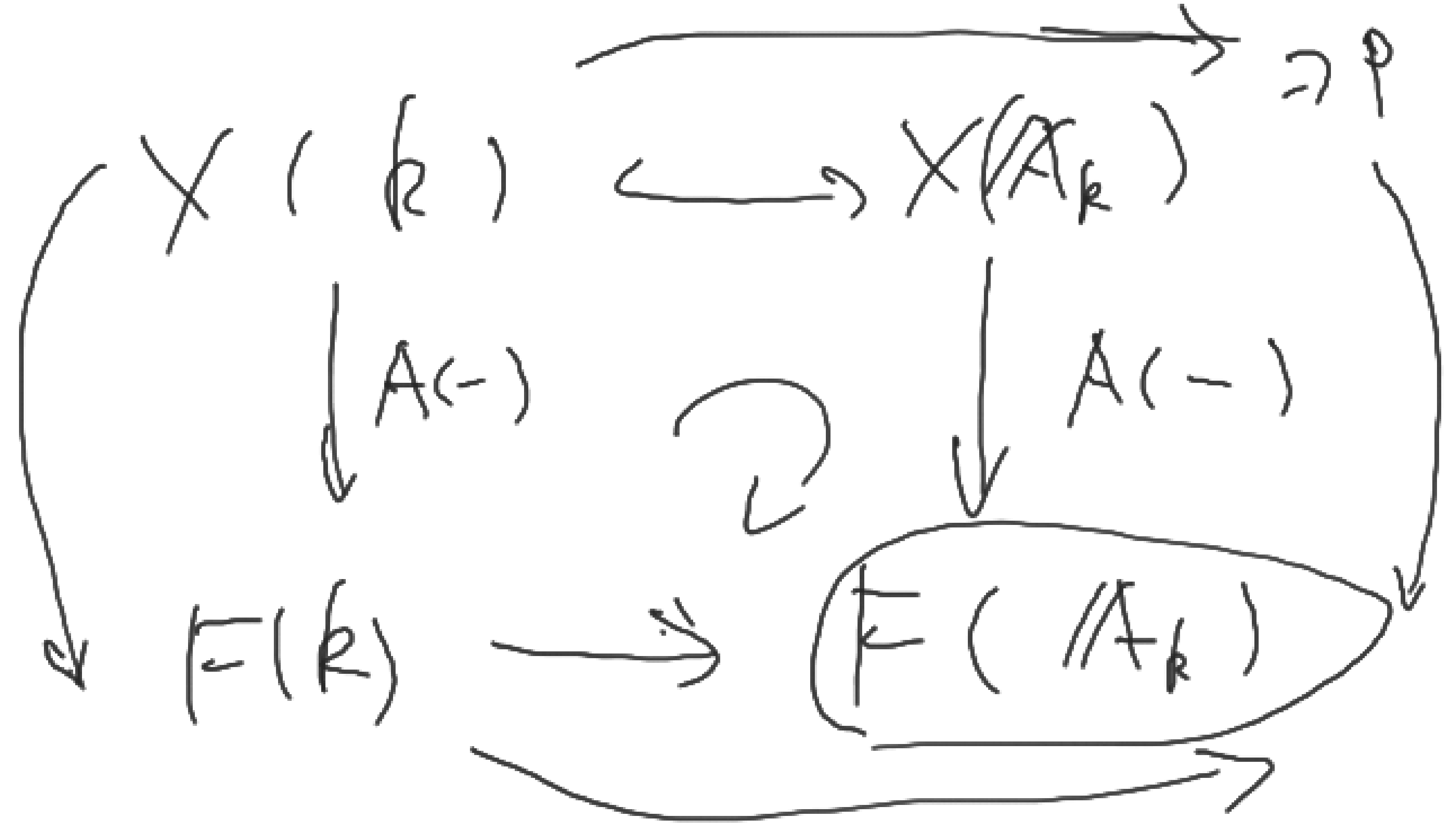
$A \in F(X)$

$$F(X) \xrightarrow{x^*} F(T)$$

$$A \longmapsto \underline{A(x)}$$

evaluation

(3.3)



$$\begin{tikzcd}
 & & p \\
X(k) \arrow[r, hook] \arrow[d, "A(-)"] \arrow[urr, bend left] \arrow[d, bend right=60] & X(\mathbb{A}_k) \arrow[d, "A(-)"] \arrow[from=ur, bend left] & \\
F(k) \arrow[r] \arrow[r, bend right] \arrow[ur, phantom, "\circlearrowright"] & F(\mathbb{A}_k) \arrow[from=uur, bend left] &
\end{tikzcd}$$

From this, if define

$A \in \mathrm{Br}(X)$

$$\begin{array}{c} X(\mathbb{A}_k)^A = \{ p \in X(\mathbb{A}_k) \mid A(p) \in \mathrm{im}(F(k) \to F(\mathbb{A}_k)) \} \\ \rotatebox{90}{$\subseteq$} \\ X(k) \end{array}$$

3.4 Def. The set $X(\mathbb{A}_k)^A$ is called the obstruction defined by $A \in F(X)$

Imposing all constraints made by $A \in F(X)$

$$X(k) \subseteq X(\mathbb{A}_k)^F = X(\mathbb{A}_k)^{F(X)} := \bigcap_{A \in F(X)} X(\mathbb{A}_k)^A .$$

called the F-set (obstruction)

$$\emptyset \subseteq X(k) \subseteq X(\mathbb{A}_k)^F \subseteq X(\mathbb{A}_k)^A \subseteq X(\mathbb{A}_k)$$

3.5 Def (i) if $X(\mathbb{A}_k) \neq \emptyset$

but $X(\mathbb{A}_k)^F = \emptyset$. we say

$\exists$ F-obs to local-global.

(ii) if $X(\mathbb{A}_k)^F \neq \emptyset \Rightarrow X(k) \neq \emptyset$

we say the F-obs is

the only one.

3.6 ~~Eg~~. $F = \mathrm{Br} = H^2_{et}(-, \mathbb{G}_m)$

BM obs

$\bigoplus_v \mathrm{Br}\, k_v \cong \mathrm{Br}(\mathrm{Spec}\, \mathbb{A}_k)$

3.7 ~~Eg~~ $F = \check{H}^1_{fppf}(-, G)$. G k-gp scheme $G \to \mathrm{Spec}\, k$

first Čech cohomology

If G commutative, $\check{H}^1_{fppf}(X, G) = H^1_{fppf}(X, G)$

k - # field. G sm. $\Rightarrow$ $\qquad \shortparallel$

$H^1_{et}(X, G)$

$$Y(k) \subseteq X(\mathbb{A}_k)^{\check{H}^1_{fppf}(-,\,\underline{G})} \subseteq X(\mathbb{A}_k)$$

3.8 Def The descent obstruction is given by

$$X(\mathbb{A}_k)^{desc} = \bigcap_{\text{all affine } k\text{-gp } G} X(\mathbb{A}_k)^{\check{H}^1_{fppf}(-,G)}$$

3.9 Prop [Sk01] For X regular, quasi-projective

$$X(\mathbb{A}_k)^{desc} \subseteq X(\mathbb{A}_k)^{Br} = \bigcap_{n \geq 1} X(\mathbb{A}_k)^{H^1_{fppf}(-,PGL_n)}$$

Can we find smaller subset?

3.10. Def. An X-torsor under an X-gp schem G is a X-scheme Y with an action of G, (compatible with the projection to X)

and s.t. ,

$$Y \xrightarrow{G} X \qquad \left(\begin{array}{l} G \ \ k\text{-gp} \\ G_X = G \times_k X \end{array} \right)$$

s.t. $$Y \times_X G \longrightarrow Y \times_X Y \quad \text{is } \underline{\text{iso}}$$

$$(y, s) \longmapsto (y, \underline{ys})$$

(By Groth's fppf descent, this is to say

Y is fppf-locally $\underline{\text{trivial}}$. i.e.

$\Rightarrow$ fppf covering $(U_i \longrightarrow X)_i$ s.t.

$$\left(Y_{U_i} = Y \times_X U_i \circlearrowleft G_{U_i} \right) \xrightarrow{\sim} \left(G_{U_i} \circlearrowleft G_{U_i} \right))$$

as pointed set

$$\check{H}^1(X, G) \xrightarrow{\sim} \mathrm{Tors}(X, G)/\cong$$

$$e \longleftrightarrow [G]$$

the $\boxed{\text{groupoid}}$ of X-torsor under G.

obj. $Y \xrightarrow{G} X$

mor.

$$\begin{tikzcd} Y \arrow[d, "s"'] \arrow[dr, "G"] & \\ Y' \arrow[r, "G"'] & X \end{tikzcd}$$

G-equivariant

If G is comm.,

$$H^1(X, G) \xrightarrow{\sim} \mathrm{Tors}(X, G)/\cong$$

$$0 \longleftrightarrow [G]$$

3.11 Prop (descent by torsors)

G k-gp. $f : Y \xrightarrow{G} X$ G-torsor, $[Y] \in \check{H}^1_{fppf}(X, G)$

$$Y(\underbrace{\mathbb{A}_k}_{U/k})^{[Y]} = X(\mathbb{A}_k)^f = \bigcup_{\sigma \in H^1(k,G)} f^\sigma(\underbrace{Y^\sigma(\mathbb{A}_k)}_{U})$$

where $f^\sigma : Y^\sigma \xrightarrow{G^\sigma} X$ is the torsor twisted by σ.

$$\left(\text{key}: \quad [Y^\sigma] \in \underline{H^1(X,G)} \atop \underline{= [Y] + [\sigma]} \right)$$

<u>3.13.</u> We have

$$\underline{X(\mathbb{A}_k)^{\text{desc}}} = \bigcap_{f: Y \xrightarrow{G} X \text{ all torsors}} \bigcup_{\sigma \in H^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k))$$

all affine G

2nd part, $$X(k) = \coprod_{\sigma \in H^1(k,G)} f^\sigma(Y^\sigma(k))$$

3.14 . Def (i) [Poonen 99, ~~$\mathcal{B}$~~ ....]

The étale-Brauer obs.

$$X(k) \hookrightarrow X(\mathbb{A}_k)^{\text{ét}, \text{Br}} = \bigcap_{\substack{\text{finite } k\text{-gp } G \\ \text{all } f: Y \xrightarrow{G} X}} \bigcup_{\sigma \in H^1(k, G)} f^\sigma\left(Y^\sigma(\mathbb{A}_k)^{\text{Br}}\right)$$

(ii) The iterated descent obs is

$$X(\mathbb{A}_k)^{\text{desc}} \supseteq X(\mathbb{A}_k)^{\text{desc}, \text{desc}} = \bigcap_{\substack{\forall \text{ affine } G \\ \forall f: Y \xrightarrow{G} X}} \bigcup_{\sigma \in H^1(k, G)} f^\sigma\left(Y^\sigma(\mathbb{A}_k)^{\text{desc}}\right)$$

3.15 Thm [Sk 09. Stoll 07. Demarch 09.

CDX 16] Let X sm,

q.p. geo. integral, k-variety. Then.

$$X(\mathbb{A}_k)^{\text{ét}, Br} = X(\mathbb{A}_k)^{desc}$$

3.16 thm. [Cao 20]. X as before,

$$X(\mathbb{A}_k)^{desc} = X(\mathbb{A}_k)^{desc, desc} = X(\mathbb{A}_k)^{desc, desc, desc} \cdots\cdots$$

Upshot: no obs smaller than descent is found

3.17 (Sk-Zahin-L) $(X \times_k Y)(\mathbb{A}_k)^{Br} = X(\mathbb{A}_k)^{Br} \times Y(\mathbb{A}_k)^{Br}$

$(-)(\mathbb{A}_k)^{Br}$. preserve fin. product.

3.18 . (Schlank-Harpaz)

$(X \times_k Y)(\mathbb{A}_k)^{et, Br} = X(\mathbb{A}_k)^{et, Br} \times Y(\mathbb{A}_k)^{et, Br}$

Homotopy type

$(-)(\mathbb{A}_k)^{et, Br}$ . . . .

$(et\, Br) = desc$

$(-)(\mathbb{A}_k)^{desc}$ . . . .

~~A~~ S–H

(étale) Homotopy obs.

- $\underline{X(\mathbb{A}_k)^{\mathbb{Z}h}} = X(\mathbb{A}_k)^{Br}$
- $\underline{X(\mathbb{A}_k)^{h}} = \underline{X(\mathbb{A}_k)^{\acute{e}t,\, Br}}$

over $\mathbb{C}$. $\quad$ $-(\mathbb{A}_k)^h$ $\underline{\text{preserve fin. prod.}}$

find more classes of X s.t.

- obs is only one . . . —— .
- find more obs (smaller from
  — descent) .
- relations between obs

X genus > 1 curve

$X(k)$ finite .