Étale homotopy obstructions

§1 Motivation

1.1. A rational pt is a section to the structure morphism.

$$\begin{tikzcd} \mathrm{Spec}\, k \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & X \arrow[r] & \mathrm{Spec}\, k \end{tikzcd}$$

$\rightsquigarrow$ More generally a map

$$f : X \to Y$$

- We may consider instead sections to topological spaces

- scheme $\underset{\text{how to ?}}{\rightsquigarrow}$ topological spaces.

  - Naive : Zariski topology $\longrightarrow$ too course.
  - Over $\mathbb{C}$: $f^{an} : X(\mathbb{C}) \longrightarrow Y(\mathbb{C})$

    (top ~~ind~~ inherited from $\mathbb{C}$)

<u>1.2</u> • For general schemes (over arbitrary fields) we use étale topology

$\rightsquigarrow$ étale <u>topos</u>

~~stank~~ <u>schlank</u> $X_{ét}$ (~~topology~~ space)

At least for $X/\mathbb{C}$ $X_{ét}$ carries a big part of info of $X(\mathbb{C})$.

(<u>eg.</u> $H^i_{ét}(X, F) \cong H^i(X(\mathbb{C}), F)$) fin. gp.

- For general topos $\mathcal{X}$.

  $\rightsquigarrow$ (pro-) homotopy type.

- Schemes $\longrightarrow$ Topos $\longrightarrow$ Homotopy type.

- Variant: relative version.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & B \end{tikzcd} \quad \rightsquigarrow \quad \underset{\text{rel. hom. type}}{|X_{et}/B_{et}| \longrightarrow |Y_{et}/B_{et}|}$$

- In order to deal with "sheaf of homotopy types" in a coherent way, we use <u>∞-topos</u>. [Lurie HTT]

1967. <u>Artin & Mazur</u>

§2 Sketch of ∞-categories

For simplicity, we ignor set theoretical issue

2.1 (2,1)-categories & pseudofunctors.

A (2,1)-Cat $\mathcal{C}$ $\begin{cases} 0\text{-cell} & \text{obj.} \quad X, Y, Z \cdots \\ (\text{morphism}) & \\ 1\text{-cell} & \text{morphism} \quad X \xrightarrow{f} Y \\ 2\text{-cell} & \text{natural iso} \end{cases}$

\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\wr"] & Y \end{tikzcd}

[SP] nlab

[Kerodon] or HTT

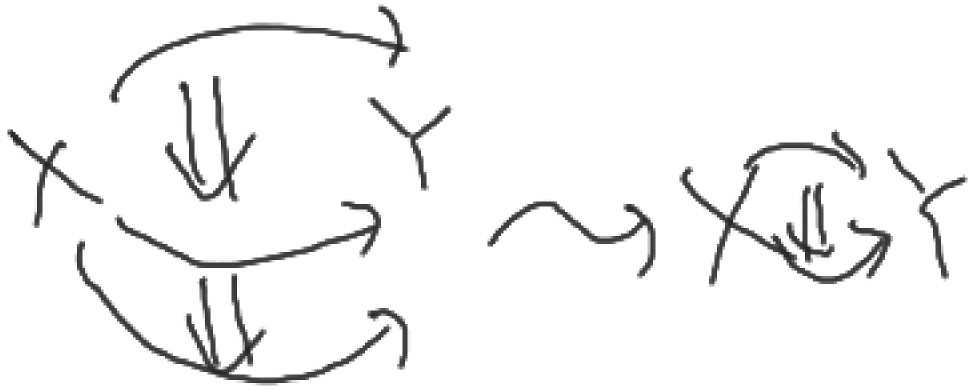
• Mapping space :

$\underline{\mathrm{Map}_e(X,Y)}$ a $\underline{\text{category}}$

obj 1-cell

mor 2-cell

:



• composition :

$$\mathrm{Map}_e(X,Y)\times \mathrm{Map}_e(Y,Z)\longrightarrow \mathrm{Map}_{\mathcal{C}}(X,Z)$$

a $\underline{\text{functor}}$

$X \xrightarrow{\mathrm{id}} X$

- Identity : id_X $\underline{\mathrm{id}_f}$
- subject to additional axioms ......
  coherency

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow \mathrm{id}"] & Y \end{tikzcd}$$

$\underline{\text{Eg}}$ $\mathcal{C}at_{/\mathcal{C}}$, $\mathcal{G}pd_{/\mathcal{C}}$, $\mathcal{T}ps_{/X}$ ......

0 cats 1 funcs 2 nat iso.

$\mathcal{A}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{D} \end{tikzcd}$$

$\mathcal{D} \to$ groupoid

$$\begin{tikzcd} X \arrow[d] & f(X) \arrow[r, "\sim"] \arrow[d] & g(X) \arrow[d] \\ Y & f(Y) \arrow[r, "\sim"] & g(Y) \end{tikzcd}$$

A pseudofunctor between two (2,1)-cats.
(or weak 2-functor)

$$F : \mathcal{C} \longrightarrow \mathcal{D}.$$

$$X \rightsquigarrow F(X)$$

$$\mathrm{Map}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(F(X), F(Y))$$

functor

($X \xrightarrow{f} Y \rightsquigarrow F(X) \xrightarrow{F(f)} F(Y)$

$$\begin{tikzcd} X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Y \end{tikzcd} \rightsquigarrow \begin{tikzcd} F(X) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & F(Y) \end{tikzcd}$$

- $$\begin{tikzcd} & Y \arrow[dr] \arrow[d, phantom, "\Downarrow \cong"] & \\ X \arrow[ur] \arrow[rr] & & Z \end{tikzcd} \longrightarrow \begin{tikzcd} & F(Y) \arrow[dr] \arrow[d, phantom, "\Downarrow \cong"] & \\ F(X) \arrow[ur] \arrow[rr] & & F(Z) \end{tikzcd}$$

- $F(id_X) \xrightarrow{\sim} id_{F(X)}$

- coherency ....

Joyal & Lurie develop ∞-cat under the model of simplicial sets.

2.3. ~~Def~~. A simplicial set is a

functor $K : \Delta^{op} \longrightarrow Set$

- obj: $[n]$. $n \geq 0$.
  $\{0, 1, \ldots, n\}$

$[m] \longrightarrow [n]$

- mor: non-decreasing maps
  (generated by face maps d_k^n
  and degeneration maps. s_k^n

$d_k^n : [n-1] \longrightarrow [n]$
inj, only missing k

$s_k^n : [n+1] \longrightarrow [n]$
surj, duplicating k and $k+1$

2.4 Eg: • Čech nerve of $X_0 \xrightarrow{f} X_{-1}$ in $\mathcal{C}$

$$\check{C}(f): \Delta^{op} \longrightarrow Set$$

$$[n] \longmapsto X_n := \underbrace{X_0 \times_{X_{-1}} X_0 \cdots \times_{X_{-1}} X_0}_{n+1}$$

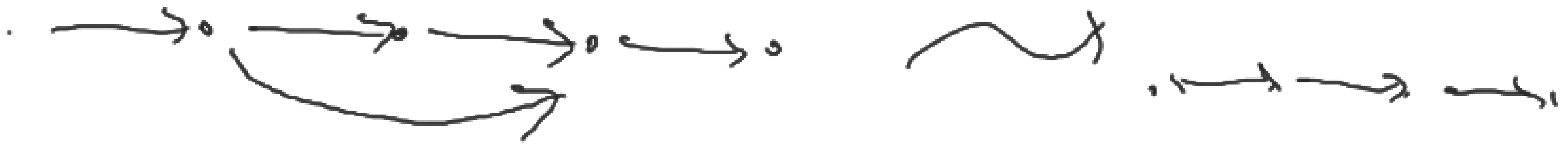
$$X_2 \; \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} \; X_1 \; \substack{\xrightarrow{p_1}\\ \longleftarrow\\ \xrightarrow[p_2]{}} \; X_0 \xrightarrow{f} X_{-1}$$

$$\begin{tikzcd} X_1 \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\square"] & X_0 \arrow[d, "f"] \\ X_0 \arrow[r, "f"'] & X_{-1} \end{tikzcd}$$

• Nerve of a cat $\mathcal{C}$.

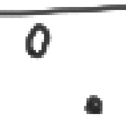
$$\underline{N(\mathcal{C})}: \Delta^{op} \longrightarrow Set$$

$$[n] \longmapsto \{ X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n \}$$



- representable simplicial sets.

$$\Delta^n = \underline{\mathrm{Map}_{\mathrm{Set}_\Delta}(-,[n])} : \Delta^{op} \to \mathrm{Set}$$

Δ^0 

$\cup\!\vert$

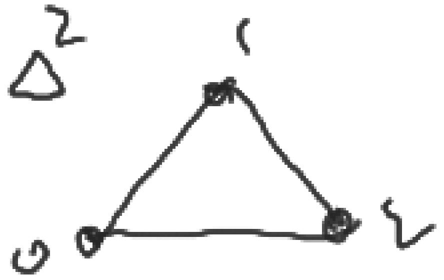
border $\partial\Delta^n$

Δ^1 

$\cup\!\vert$

i-th horn Λ^n_i

Δ^2 

$\partial\Delta^2$ 

Λ^2_1 

Yoneda. $\rightsquigarrow$ $K: \Delta^{op} \to$ Set

$K_n := K([n])$ amounts to

maps $\underline{\Delta^n} \to K$.

$n = 0, 1, 2$

vertix edge, 2-simplices ...

- Set_Δ is Cartesian closed

$\rightsquigarrow$ define. $\underline{Map(X, Y)}: \Delta^{op} \to$ Set

$[n] \longmapsto \underset{Set_\Delta}{Map}(\Delta^n \times X, Y)$

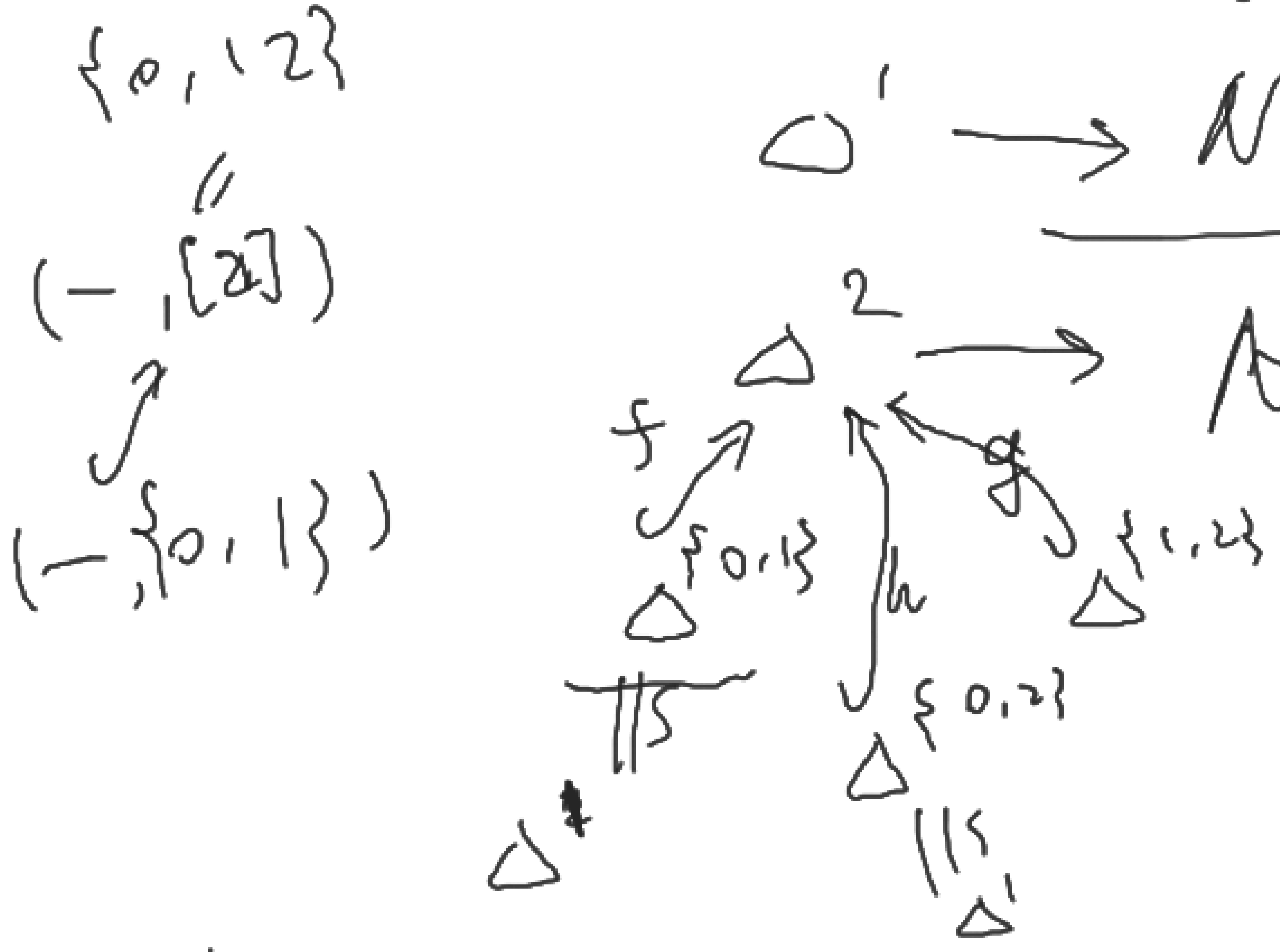
$X, Y \in Set_\Delta$.

nerve $N(\mathcal{C})$ $\mathcal{C}$ 1-cat

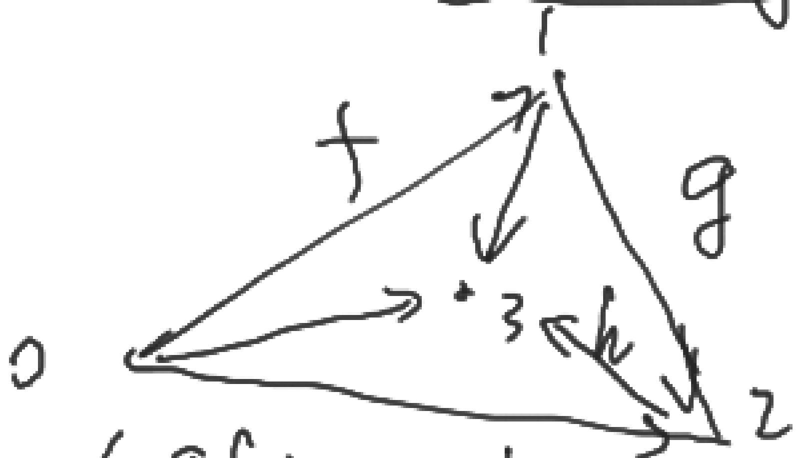
$\Delta^0 \longrightarrow N(\mathcal{C})$ are objects of $\mathcal{C}$

$\Delta^1 \longrightarrow N(\mathcal{C})$ morphism.

$\Delta^2 \longrightarrow N(\mathcal{C})$



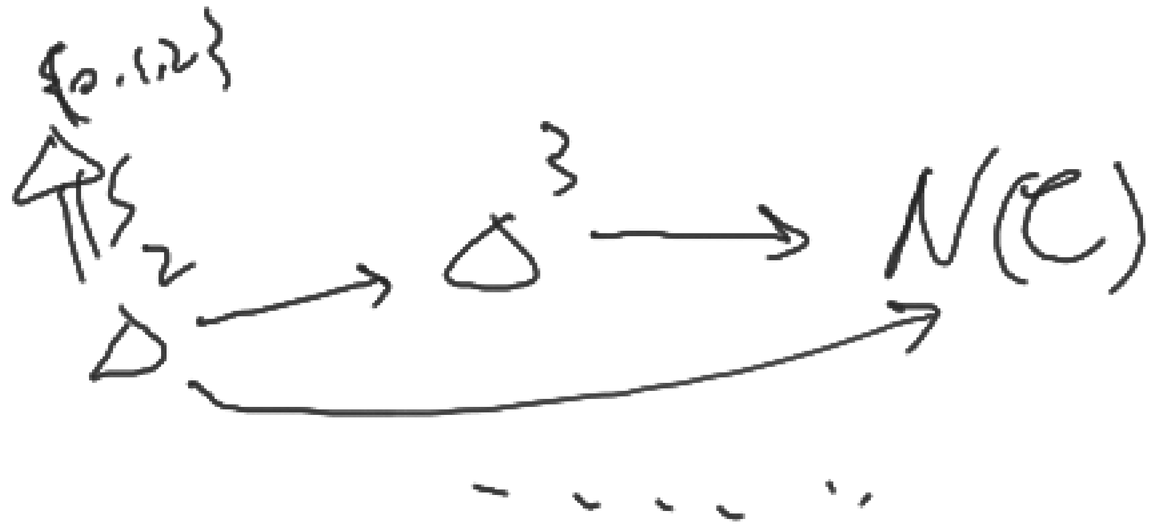
$f \circ g = h$



$h(gf) = (hg)f$

face $\longleftrightarrow$ homotopy witness of composition

interior $\longleftrightarrow$ of associa



2.5. Def. (Joyal)

(1). A ∞-cat is a simplicial set

$$\mathcal{C} : \Delta^{op} \longrightarrow \mathrm{Set} \quad \text{s.t.}$$

$\forall\ 0 < i < n$, $\forall\ \Lambda^n_i \longrightarrow \mathcal{C}$, $\exists$ a dotted arrow rendering the diagm comm.

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\
\Delta^n \arrow[ur, dotted, "\exists"] &
\end{tikzcd}$$

(2) replacing $0 < i < n$ with $0 \le i \le n$, we obtain the notion of a Kan complex (or ∞-groupoid)

$$\begin{tikzcd} & 1 \arrow[dr, "g"] & \\ 0 \arrow[ur, "f"] \arrow[rr, "id"'] & & 2 \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dr, "f"] & \\ 0 \arrow[ur, "g'"] \arrow[rr, "id"'] & & 2 \end{tikzcd}$$

2.6 Prop (HTT) A simplicial set K is iso to a nerve (of some cat) iff the extension in 2.5 (1) is unique

Schlank
Etale homotopy and obstructions to rational points.