2.5 Def

∞-cat is a simplicial set

$\mathcal{C} : \Delta^{op} \longrightarrow \mathrm{Set}$ s.t.

$\forall\ 0 < i < n$, $\forall\ \Lambda^n_i \longrightarrow \mathcal{C}$, $\exists$ a lift

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

$n = 3$

- $0 \le i \le n$. $\mathcal{C}$ is a ∞-groupoid
  Kan complex
- $\mathcal{C} \cong N(C) \iff \exists!$ lift.

2.7 Homotopy cats

The nerve gives fully-faithful embedding

$$N : \mathcal{C}at_1 \longrightarrow \mathcal{C}at_\infty$$

and admits a left adj

$$h : \mathcal{C}at_\infty \longrightarrow \mathcal{C}at_1$$
$$\mathcal{C} \longmapsto h\mathcal{C}$$

homotopy cat of $\mathcal{C}$

Explicitly, $h\mathcal{C}$ has the same objs as $\mathcal{C}$, but $\mathrm{Map}_{h\mathcal{C}}(X, Y) = \pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y)$

$= \mathrm{Map}_{\mathcal{C}}(X, Y) / \text{homotopy} \sim \qquad X, Y \in \mathrm{Ob}\,\mathcal{C}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\mathrm{ob}\, h\mathcal{C}}$

$\mathrm{Map}_{\mathcal{C}}(X, Y)$:

$$\begin{tikzcd} \Delta^{\{1\}} \arrow[dr] \arrow[drr, bend left, "Y"] & & \\ & \Delta^{1} \arrow[r] & \mathcal{C} \\ & \Delta^{\{0\}} \arrow[u, hook] \arrow[ur, "X"'] & \end{tikzcd}$$

$f \sim g \longleftrightarrow \exists$ 2-simplex $\Delta^2 \longrightarrow \mathcal{C}$ s.t.

$$\begin{tikzcd} & \bullet \arrow[dr, "id"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "g"'] & & \bullet \end{tikzcd}$$

∞-cat $\quad \mathcal{C} : \Delta^{op}_{(?)} \longrightarrow \mathrm{Set}.$

$\mathcal{C}_0 = \mathcal{C}([0]) \longleftrightarrow \{\Delta^0 \longrightarrow \mathcal{C}\}$ $\qquad$ vertices / objects

$\mathcal{C}_1 \longleftrightarrow \{\Delta^1 \longrightarrow \mathcal{C}\}$ $\qquad$ edge / morphism

2.9 Homological algebra.

- R be a ring. Lurie construct an ∞-cat $\mathcal{D}_\infty(R)$ s.t.
  - $h\mathcal{D}_\infty(R) = D^+(R)$
  - Vertices of $\mathcal{D}_\infty(R)$ — complexes of proj. R-mod bounded below
  - 1-simplices - - - - - - — chain maps $X \to Y$
  - 2-simplex - - - - - - — $X \xrightarrow{f} Y \xrightarrow{g} Z$, $X \xrightarrow{h} Z$ and a homotopy $g \circ f \sim h$

- $\mathcal{S} = \mathrm{Grpd}_\infty$, "$\infty$-cat of spaces" ~~Grpd~~ Grpd_1

$\mathcal{C}$ — ∞-cat admitting all colimits,

$c_0 \in \mathcal{C}$. define a functor

$\underline{c_0 \otimes -} : \mathcal{S} \longrightarrow \mathcal{C}$.

$$X \longmapsto \operatorname*{colim}_{x \in X} \mathrm{const}(c_0) \text{ where}$$

$$\mathrm{const}(c_0) : X \xrightarrow{x \longmapsto c_0} \mathcal{C} \text{ is the constant functor on } c_0.$$

- View $R \in D_\infty(R)$ as an obj concentrated on degree 0. $\rightsquigarrow$ $R \otimes - : \mathcal{S} \to D_\infty(R)$

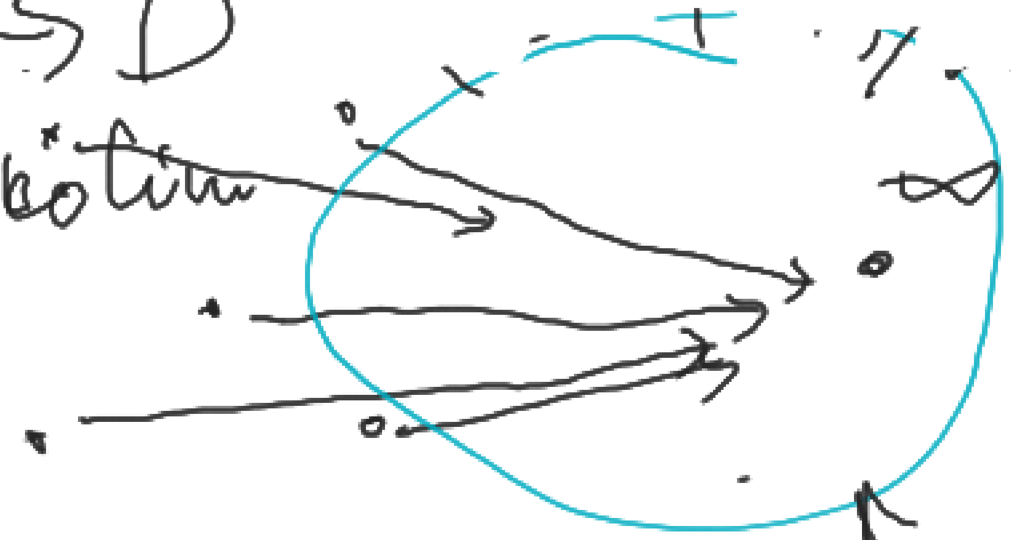
HTT.

$F: \mathcal{C} \longrightarrow \mathcal{D}$ a functor

right cone of $\mathcal{C}$

colimit $\tilde{F}: \mathcal{C}^{\triangleright} \longrightarrow \mathcal{D}$

$v \longmapsto$ colim



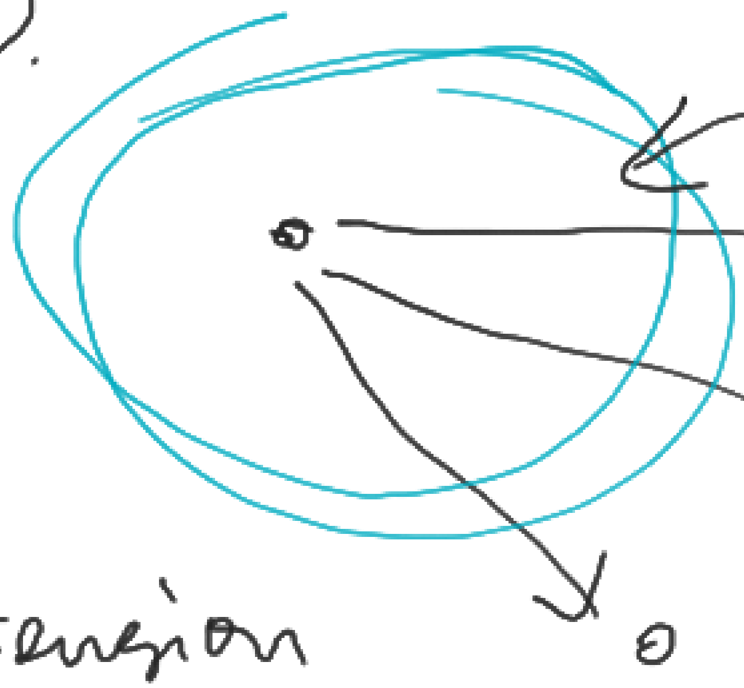
universal property

$\Updownarrow$

a initial object of

$\mathcal{D}^{\triangleleft}$

Limit



a final obj of

Kan extension

§3. ∞-topoi & étale homotopy type.

3.1 ~~Def~~. (∞-topoi)

- Let $\mathcal{C}$ be a (small) ∞-cat.

$$\underline{\mathrm{Psh}_\infty(\mathcal{C})} := \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$$

(compare $\mathrm{Psh}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$
for ordinary cat $\mathcal{C}$)

- An ∞-topos is an ∞-cat $\mathcal{X}$ that is a left exact localization of a presheaf ∞-cat.

(i.e. $\exists$ small ∞-cat $\mathcal{C}$, s.t.

$$\mathcal{X} \xrightarrow[\text{sheafification}]{} \mathrm{Psh}(\mathcal{C})$$ has a left exact (left adj.)

(In ordinary sense, $\mathcal{X}$ is a topos if $\mathcal{X}$ is equiv. to $\mathrm{Shv}(\mathcal{C})$ for some site $\mathcal{C}$. eg. X_{et}

• $\underline{Tps_\infty}$ — obj: ∞-topos.

mor. $\mathcal{X} \longrightarrow \mathcal{Y}$

$f^* : \mathcal{Y} \rightleftarrows \mathcal{X} : f_*$ s.t.

f^* is left exact.

ét cohomolog:

$X \longrightarrow Y$ of sch.

$\underline{f_* : Ab(X) \longrightarrow Ab(Y)}$

$\underline{f^*} : Ab(Y) \longrightarrow Ab(X)$

$Rf_* : D^+(X) \longrightarrow D^+(Y)$

$\cancel{f_*}$ $f^* : D^+(Y) \to D^+(X)$

$Rf_!$ Lf^*

$\mathcal{I}_\infty$ has $\mathcal{S}$ as a final

obj: $c: \mathcal{X} \longrightarrow \mathcal{S}$

$c_* : \mathcal{X} \longrightarrow \mathcal{S}$ "$\underline{\text{global section}}$"

$c^* : \mathcal{S} \longrightarrow \mathcal{X}$ "$\underline{\text{constant sheaf}}$"

§4. shape functor & étale homotopy obstructions.

4.1. Def. (Pro cat) (1) Let $\mathcal{C}$ be ∞-cat. then $\exists$ $\mathrm{Pro}(\mathcal{C}) \in \mathrm{Cat}_\infty$. and a ful. faith. functor $j : \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$

s.t. – $\mathrm{Pro}(\mathcal{C})$ admits small cofiltered limits.

– If $\mathcal{D}$ - - -

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr] & \mathrm{Pro}(\mathcal{C}) \arrow[d, dashed] \\ & \mathcal{D} \end{tikzcd}$$

pro cat of $\mathcal{C}$.

- $\mathcal{C}$ small. $\underline{\mathrm{Pro}(\mathcal{C})} \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{S})^{op}$

spanned by $\underset{\text{cofil.}}{\lim} \underline{\mathrm{Map}_{\mathcal{C}}(X, -)}$, $X \in \mathcal{C}_0$.

<u>eg</u>. $\mathcal{C} = \left(\text{finite groups}\right)$, $\mathrm{Pro}(\mathcal{C}) = \left(\text{profinite gps}\right)$.

<u>4.3 Def</u>. (Shape). By universal $\underline{\mathrm{Pro}(\mathcal{C})}$.

$$\rightsquigarrow \quad \mathcal{S}_{/-} : \mathcal{S} \longrightarrow \underline{\mathcal{T}\mathrm{ps}_\infty}$$
$$X \longmapsto \mathcal{S}_{/X}$$

induces $\quad \mathcal{S}^{pro}_{/-} : \mathrm{Pro}(\mathcal{S}) \longrightarrow \mathrm{Tops}_\infty$

$\qquad\qquad\qquad\qquad j(\mathcal{X}) \longmapsto \mathcal{S}_{/\mathcal{X}}$

admitting a left adj

$\mathrm{Sh} : \mathrm{Tops}_\infty \longrightarrow \mathrm{Pro}(\mathcal{S})$

$\mathcal{X} \longmapsto \mathrm{Sh}(\mathcal{X}) = |\mathcal{X}|$

or $|-|$ $\qquad$ called shape of functors

(2). If $\mathcal{X} \in \mathrm{Tops}_\infty$. $\mathcal{X} \xrightarrow{c} \mathcal{S}$

$\rightsquigarrow \quad c_! : \underset{\mathrm{Fun}(\mathcal{X}, \mathcal{S})^{op}}{\mathrm{Pro}(\mathcal{X})} \xrightarrow{\text{comp. } c^*} \underset{\mathrm{Fun}(\mathcal{S}, \mathcal{S})^{op}}{\mathrm{Pro}(\mathcal{S})} \rightsquigarrow |\mathcal{X}| = c_!(*_{\mathcal{X}})$

$= c_* c^* : \mathcal{S} \to \mathcal{S}$

(3). (relative)

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "t"'] & & Y \arrow[dl] \\ & B & \end{tikzcd}$$ in $\mathcal{T}ps_\infty$.

$\rightsquigarrow \quad |X/B| = t_!(*) = c_* t^* : B \longrightarrow \mathcal{S}$

$\rightsquigarrow \quad \bar{f} : |X/B| \longrightarrow |Y/B|$

"$*$"

In part., $\underline{B = Y}$, $|Y/Y| = *_Y$

think $|X/Y|$ as a "sheaf".

(sections of $\bar{f}$) amounts to a "global section"

(4). (linearised). Define a ∞-cat

$$\mathcal{X}_R := \mathrm{Fun}_{\lim}(\mathcal{X}^{op}, \mathcal{D}_\infty(R)).$$

Recall $R \otimes - : \mathcal{S} \longrightarrow \mathcal{D}_\infty(R)$.

$$\rightsquigarrow \quad R \otimes - : \mathcal{X} \longrightarrow \mathcal{X}_R.$$

$$\rightsquigarrow \quad |\mathcal{X}/\mathcal{Y}|_R := R \otimes |\mathcal{X}/\mathcal{Y}| \in \mathrm{Pro}(\mathcal{Y}_R)$$

$R = \mathbb{Z}$.

4.4. Étale homotopy type

- (~~∞~~ -(small) étale topos)

$X \in Sch$. $\underline{X_{\acute{e}t}} := PSh({\acute{e}t}/X)[W^{-1}]$

where W are morphisms $F \to G$

s.t. $\forall$ geo. pt $x \to X$

$x^* F \longrightarrow x^* G \in S$ is an equiv.

- Functor $\Pi_\infty^{\acute{e}t}$: $Sch \xrightarrow{\acute{e}t} Tops_\infty \xrightarrow[|\cdot|]{Sh} Pro(S)$

<u>homotopy type</u>. $X \longmapsto |X_{\acute{e}t}|$

$\circ \quad \left|X_{et}/Y_{et}\right| \qquad \left|X_{et}/Y_{et}\right|_R .$

<u>4.5</u> Homotopy obs. set.

$$X(k^h) := \pi_0 \Big(\mathrm{Map}_{\mathrm{Pro}(k)} \big(\underbrace{|k_{et}/k_{et}|}^{\cong *}, |X_{et}/k_{et}| \big) \Big)$$

global section.

$$X(k^{h\mathbb{Z}}) := \pi_0 F \quad \text{where}$$

$$\begin{tikzcd}
F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{id} \arrow[d] \\
\mathrm{Map}_{\mathrm{Pro}(k_{\mathbb{Z}})}\big(\underbrace{|k/k|_{\mathbb{Z}}}_{=\mathbb{Z}}, |X/k|_{\mathbb{Z}}\big) \arrow[r] & \mathrm{Map}_{\mathrm{Pro}(k)}\big(\underbrace{|k/k|_{\mathbb{Z}}}_{=\mathbb{Z}}, |k/k|_{\mathbb{Z}}\big)
\end{tikzcd}$$

4.6 obs. local - global

$$
\begin{tikzcd}
X(k) \arrow[r] \arrow[rr, bend left] \arrow[dr] & X(k^h) \times_{X(\mathbb{A}_k^h)} X(\mathbb{A}_k) \arrow[r, "\lambda"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X(\mathbb{A}_k) \arrow[d] \\
& X(k^h) \arrow[r] & X(\mathbb{A}_k^h)
\end{tikzcd}
$$

define $X(\mathbb{A}_k)^h =$ im λ.

$\rightsquigarrow \quad X(k) \subseteq X(\mathbb{A}_k)^h \subseteq X(\mathbb{A}_k)$.

similarly, replacing h by $h\mathbb{Z}$.

$\longrightarrow \quad X(k) \subseteq X(\mathbb{A}_k)^{h\mathbb{Z}} \subseteq X(\mathbb{A}_k)$

4.7 Thm (Harpaz – Schlank 14)

X var over # field k,

geo. conn. sm. then

- $\underline{X(\mathbb{A}_k)^{h\mathbb{Z}} = X(\mathbb{A}_k)^{Br}}$
- $X(\mathbb{A}_k)^{h} = X(\mathbb{A}_k)^{\acute{e}t, Br}$ $\underline{\left(\overset{\text{geo. ind q-p}}{=} X(\mathbb{A}_k)^{desc} \right)}$ ✓
- $-(\mathbb{A}_k)^{h}$ preserve fin. prod.

  i.e. $X(\mathbb{A}_k)^{desc} \times Y(\mathbb{A}_k)^{desc} = (X \times Y)(\mathbb{A}_k)^{desc}$