

# Brief review of algebraic spaces / stacks

§1 . Motivation. (learned from Olsson  
Li, Ciarand ...).

1.1 Def . Moduli spaces for moduli problems

M :  $\text{Sch}^{\text{op}} \rightarrow \text{Set} \quad \tau \mapsto (\text{iso classes of}$   
 $\text{geo. objects} / \tau)$

- fine moduli space :

$$h_X \xrightarrow{\sim} M.$$

$X$

$$h_X = \text{Hom}(-, X).$$

- coarse moduli space  
(a)  $M \rightarrow h_X$  universal.

$$\left( \begin{array}{ccc} M & \longrightarrow & h_X \\ & \searrow & \downarrow \cong \\ & & h_Y \end{array} \right)$$

$$(b) M(k) \xrightarrow{\sim} h_X(k) = X(k)$$

for all alg. closed field  $k$ .

1.2 Thm (FGA) Let  $X$  be a ~~S~~  $S$ -scheme.

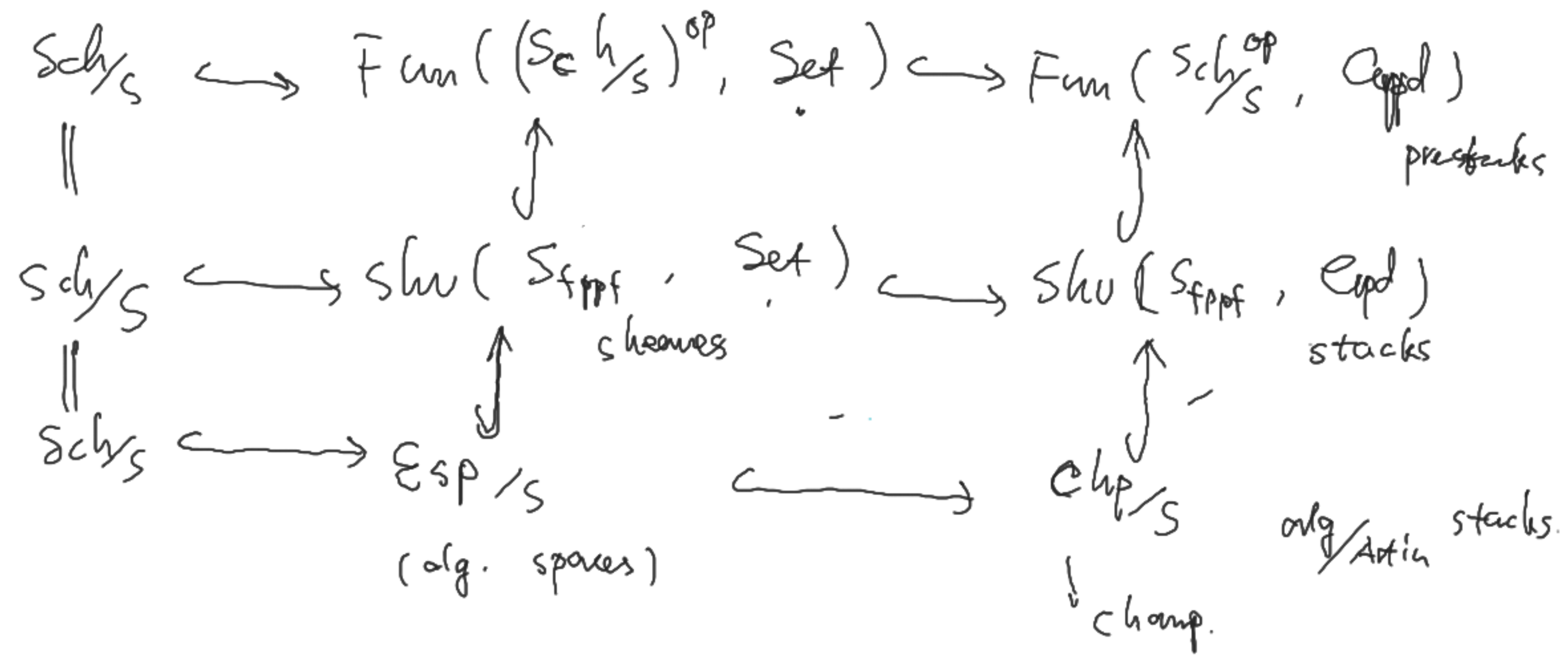
The functor  $h_X : (\text{Sch}/S)^{\text{op}} \rightarrow \text{Set}$   
 $T \longmapsto X(T)$   
" "  
 $\text{Hom}_S(T, X)$ .

is a sheaf for fppf topology. i.e.

$h_X \in \text{Shw}(S_{\text{fppf}})$   
fppf, ét. Zar.

1.3. The picture using language of functors.

Let  $S = \text{Spec } \mathbb{Z}$  be a scheme.



1.4. First example. (Moduli of elliptic curves)

$\mathcal{M}_{1,1} : T \longmapsto$  (groupoid of ell curves /  $T$ )

is an algebraic stack over  $\text{Spec } \mathbb{Z}$ .  
(sw, sep. DM)

The  $j$ -invariant induces a map

$$\hat{j} : \mathcal{M}_{1,1} \longrightarrow \mathbb{A}^1$$

In general, functors arising in moduli theory  
are of the form

$$M : \text{Sch}^{\text{op}} \longrightarrow \text{Set}$$

$$T \longmapsto \left( \frac{\text{iso class of}}{\text{certain geo. obj. of } T} \right)$$

can be lifted to

$$\mathcal{M} : \text{Sch}^{\text{op}} \rightarrow \text{Grpd}$$

$\mathcal{T} \hookrightarrow$  (groupoid of the geo. objs)

By mod iso we obtain  $\mathcal{M} \rightarrow \mathcal{M}$

1.5. (Cerberus)

$$\underline{H^1(e, \mathcal{F})} \xrightarrow{\sim} \text{Tor}_s(e, \mathcal{F}) / \text{iso}$$

groupoid of  $\mathcal{F}$ -torsors

What - about

$$H^2(e, \mathcal{F})$$

$\downarrow \mathcal{S}$

$$\text{Cerb}(e, \mathcal{F}) / \text{equiv.}$$

2-cat of gerbs (a kind of stack)

$$\begin{array}{c} e. \\ x \in \mathcal{E} \end{array} \begin{array}{c} \xrightarrow{\mathcal{F}} \\ \mathcal{Y} \end{array} \begin{array}{c} \text{sheaf} \\ \text{of gp} \end{array}$$

$$\text{Fix}_{h_x} \mathcal{Y} \rightarrow \mathcal{Y}$$

1.6. (Failed to be a sheaf)

$$\underline{H^1(-, \mathcal{G})} : (\text{Sch}/S)^{\text{op}} \longrightarrow \text{Set}$$

$\mathcal{G}$       s-grp schm

$$X \longmapsto \underline{H^1(X, \mathcal{G}_X)} = \text{Tors}(X, \mathcal{G}_X) / \text{iso}$$

is not a sheaf. (eg.  $\mathbb{F} \in \text{Ab}(\text{Set})$ )

$$\mathcal{H}^n(\mathbb{F}) := H^n(-, \mathbb{F}) : \text{Set}^{\text{op}} \longrightarrow \text{Ab}$$

$- \neq 0$

For any  $n \geq 1$ .  $\mathcal{H}^n(\mathbb{F})^+ = 0$

$$\rightsquigarrow B\mathcal{G}^- : (\text{Sch}/S)^{\text{op}} \longrightarrow \text{Epd}$$

is a sheaf in groupoid (ie. a stack)

## §2. Descent & sheaves

2.1. Recall that the objs in  $\text{Shv}(\mathcal{C})$  are those (product preserving functors)

$$F : \mathcal{C}^{\text{op}} \longrightarrow \text{Set} \quad \text{s.t.}$$

$$F(U_{-1}) \xrightarrow{\sim} \text{Lim}(F(U_0)) \rightrightarrows F(U_{-1})$$

for all covering  $U_0 \rightarrow U_{-1}$   
(with  $U_{-1} = \coprod_{U_0} U_0$ )

2.2. Def. (Descent, Zheng) Let  $f: X_0 \rightarrow X_{-1}$

be an edge of  $\mathcal{C}$  ( $\infty$ -cat)

$F: \mathcal{C}^{op} \rightarrow \mathcal{D}$ . (functor of  $\infty$ -cat).

Each nerve,  $X_\bullet: \mathcal{N}(\Delta_+)^{op} \rightarrow \mathcal{C}$ .

of  $f$ .  $[n] \mapsto \underbrace{X_0 \times \dots \times X_0}_{X_{-1} \times \dots \times X_{-1}}_{n+1}$

$f$  is of F-descent if

$F \circ X_\bullet^{op}: \mathcal{N}(\Delta_+) \rightarrow \mathcal{D}$  is a limit

diagram.

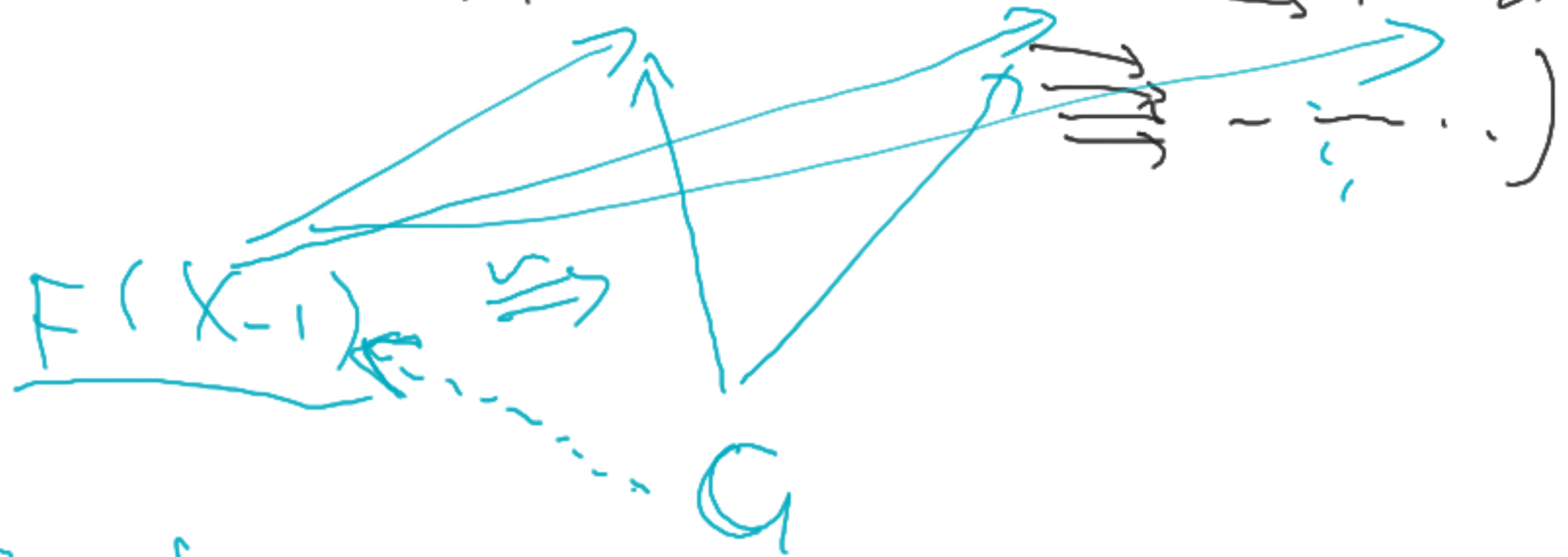
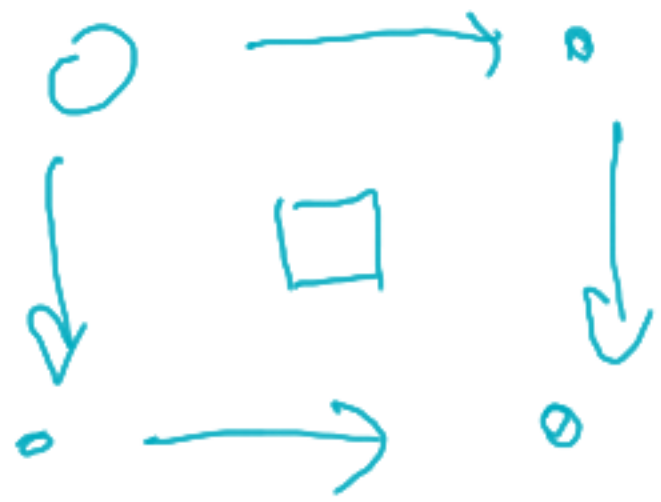
$[-1] \cong \emptyset$   $[n] = \{0, 1, \dots, n\}$



1'2.

$$F(X_{-1}) \xrightarrow{\eta} \lim_{\substack{\leftarrow \\ U \in \mathcal{O}} } F(X_U) \\ =$$

$$\lim (F(X_0) \rightrightarrows F(X_1) \rightrightarrows F(X_2) \dots)$$



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descent datum.

### 2.3. Def (Stacks)

Let  $S$  be a scheme.

A functor  $\mathcal{X} : (\text{Sch}/S)^{\text{op}} \rightarrow \text{Set}$  is a stack over  $S$  if  $\mathcal{X}$  preserves products and

every fpqc morphism  $f$  in  $\text{Sch}/S$  is of

$$f: U_0 \rightarrow U_1$$

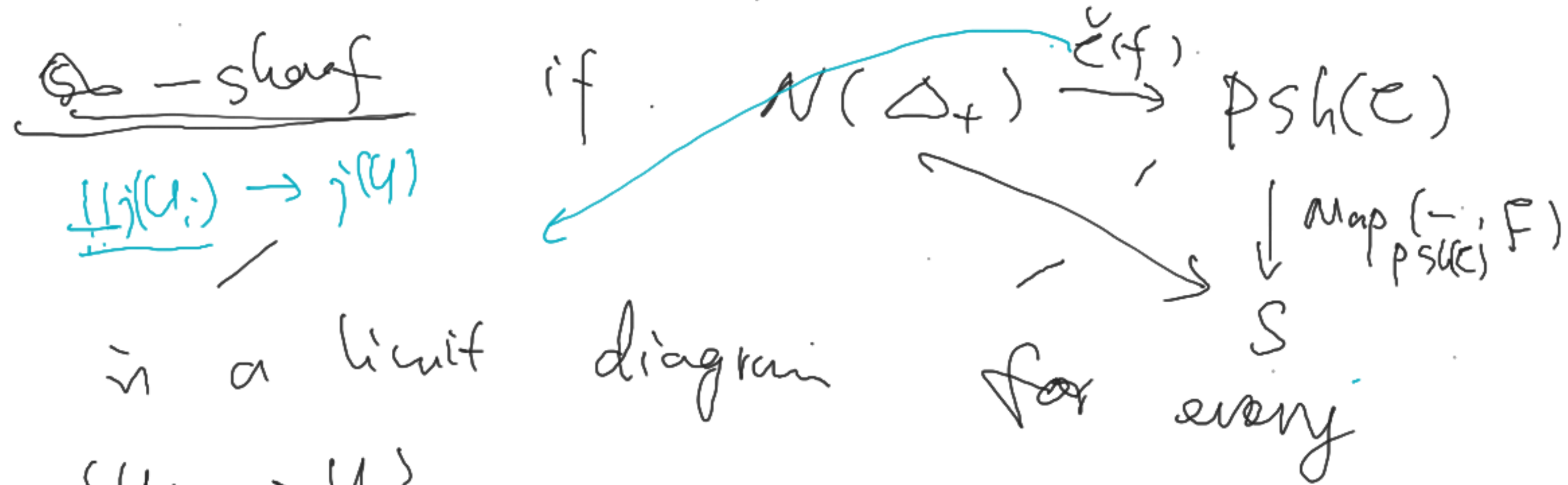
$\mathcal{X}$  is descent (i.e.

$$\mathcal{X}(U_1) \xrightarrow{\sim} \lim_{\leftarrow} (\mathcal{X}(U_0) \rightrightarrows \mathcal{X}(U_1) \rightrightarrows \mathcal{X}(U_2))$$

pp  $(U_0 \rightarrow U_1)$  descent datum

2.4 Rank.  $\mathcal{O}$ .  $\mathcal{P}$  is general. if the  $\infty$ -cat  
 $\mathcal{C}$  is equip with cross top  $\tau$ .

$F \in \text{Fun}(\mathcal{C}^\infty, \mathcal{S}) = \text{PSH}(\mathcal{C})$  is



$j$  Towels

$$\coprod_j (U_i) \rightarrow j(U)$$

is a limit

$$\{U_i \rightarrow U\}_i$$

diagram

$$\text{Shv}_e(\mathcal{C}_e) \hookrightarrow \text{PSH}(\mathcal{C})$$

one can also use sites to describe sheaves.

$$(\text{stacks} / \mathcal{S}) \cong \text{Shv}(N(\mathcal{S}_{\text{ppf}}))$$

$$\cong N(\text{Shv}(\mathcal{S}_{\text{ppf}}, \text{Cpdt}))$$

② (Groth. construction)

one can also use fibered cats. to describe stacks.

$$\chi: \text{Sch}^{\text{op}} \rightarrow \text{Cpdt} \quad \left\langle \begin{array}{c} \text{wavy arrow} \\ \text{fibered cat} \end{array} \right\rangle \quad \chi \rightarrow \text{Sch}$$

$$[\text{HTT}] \quad \text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat}_{\infty}) \begin{array}{c} \xrightarrow{\text{can.}} \\ \xleftarrow{\text{st.}} \end{array} (\text{Cat}_{\infty})_{/ \mathcal{C}}^{\text{cart}}$$

fibered cat in groupoid

§ 3.

## Representability

3.1

### Lemma (2-Yoneda)

$\mathcal{A}$

$\text{Sch}/S$

$\longrightarrow$

$\text{Fun}(\text{Sch}/S^{\text{op}}, \underline{\text{Gpd}})$

$(1$

$\longmapsto$

$h_U$

(also denoted  
by  $U$ )

be viewed as a stack and

$X \in \text{Fun}(\text{Sch}/S^{\text{op}}, \underline{\text{Gpd}})$ . Then natural eqn.

of  $\underline{\text{Gpd}}$

$X(U)$

$\xrightarrow{\sim}$

$\text{Fun}(h_U, X)$ .

3.2 Prop. All cats in diagram 1.3

admits pullbacks.

3.3 Def.

A morphism  $f: Y \rightarrow X$  of

$\text{shv}(\mathcal{S}_{\text{top}}, \text{Set})$   
sheaves

(resp.

$\text{shv}(\mathcal{S}_{\text{finf}}, \text{Cat})$   
stacks)

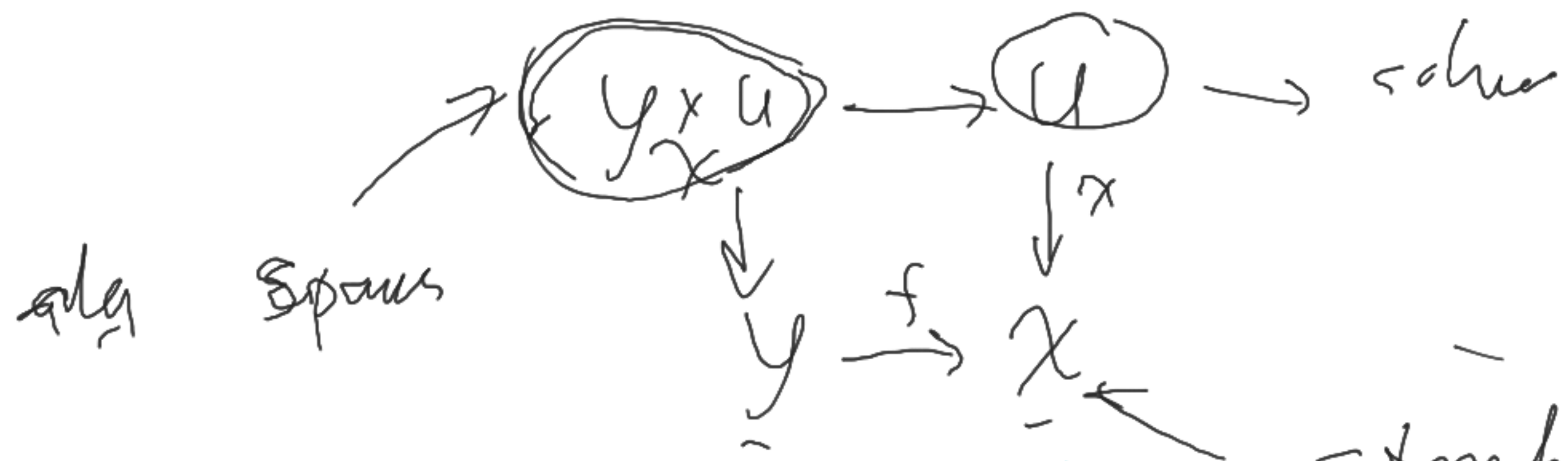
is representable if

for every  $U \in \text{Sch}/S$

and  $\forall x \in X(U)$

(by 3.1, this amounts to give a morph

$Y \times_U U \in \text{Sch}/S$  (resp.  $\mathcal{S}_{\text{top}}/S$ )  
(resp.  $\mathcal{S}_{\text{finf}}/S$ )  $U \rightarrow X$ )



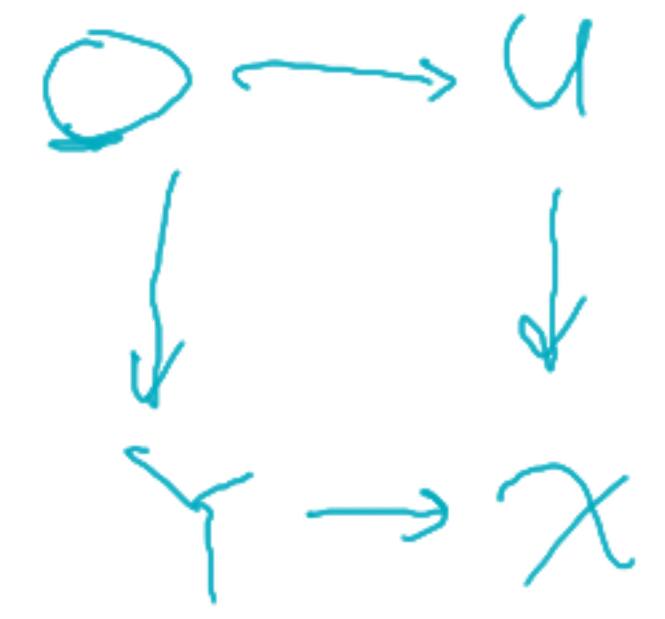
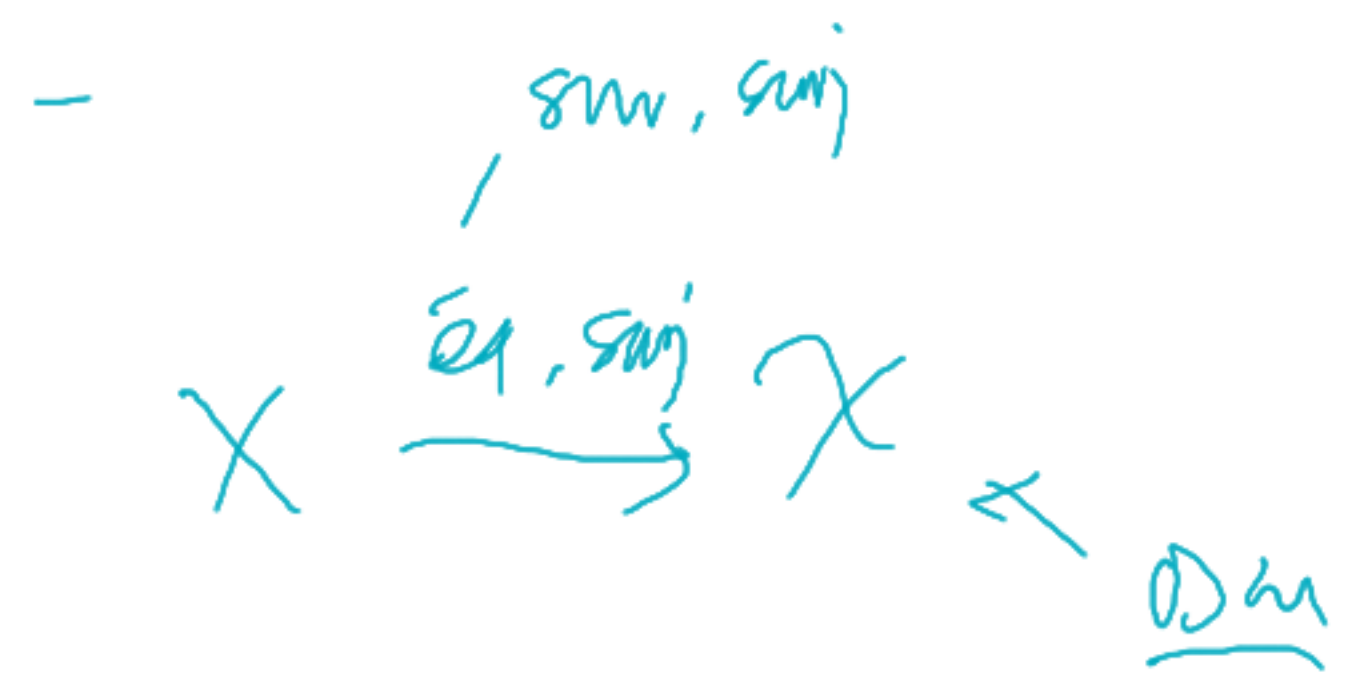
alg spans

LZ Six

operations on higher Antiu stacks and

stack

see then . . . .



$$U \xrightarrow{f} V$$

[SP,

Olsson 16 stack

HTT,

$$X_V \xrightarrow{f^*} X_U$$

Fibered cat

$\text{Kno}(\text{Sch}^{\text{op}}, \text{Cpd})$



$$(x, u) \mapsto u$$

$x \in X(u)$

Li Shizhong [139]

Cartesian

~~$\text{Sch}^{\text{op}}$~~

only space

Sch fppf

Sch<sub>ét</sub>

Stack



