Brief review of algebraic spaces / stacks

§1. Motivation (Learned from Olsson, Li, Girand ...).

1.1 ~~Ref~~. Moduli spaces for moduli problems

$\underline{M}$: $Sch^{op} \longrightarrow Set$ $\quad T \longmapsto$ (iso classes of geo. objes / T)

- fine moduli space: $\quad X \quad \underline{h_X} = Hom(-, X)$

$$h_X \xrightarrow{\sim} M.$$

- coarse moduli space

(a) $M \longrightarrow h_X$ universal. $\left(\begin{tikzcd} M \arrow[r] \arrow[dr] & h_X \arrow[d, dashed, "\exists !"] \\ & h_Y \end{tikzcd} \right)$

(b) $M(k) \xrightarrow{\sim} h_X(k) = X(k)$ for all alg. closed field k.

1.2 Thm (FGA) Let X be a S-scheme.
the functor

$$h_X : (Sch_{/S})^{op} \longrightarrow Set$$
$$T \longmapsto X(T) = Hom_S(T, X).$$

is a sheaf for fpqc topology. i.e.

$$\underline{h_X} \in Shv(S_{fpqc}).$$

fppf, ét. Zar.

1.3. The picture using language of functors.

Let S be a scheme. $(\operatorname{Spec}\mathbb{Z})$

$$\begin{tikzcd}
Sch/S \arrow[r, hook] \arrow[d, equal] & \operatorname{Fun}((Sch/S)^{op}, Set) \arrow[r, hook] & \operatorname{Fun}(Sch/S^{op}, Grpd) \\
Sch/S \arrow[r, hook] \arrow[d, equal] & \operatorname{Shv}(S_{fppf}, Set) \arrow[u, hook] \arrow[r, hook] & \operatorname{Shv}(S_{fppf}, Grpd) \arrow[u, hook] \\
Sch/S \arrow[r, hook] & Esp/S \arrow[u, hook] \arrow[r, hook] & Chp/S \arrow[u, hook]
\end{tikzcd}$$

prestacks

sheaves

stacks

(alg. spaces)

champ.

alg/Artin stacks.

1.4. First example. (Moduli of elliptic curves)

$\mathcal{M}_{1,1} : T \longmapsto$ (groupoid of ell curves $/T$).

is an algebraic stack over Spec $\mathbb{Z}$

(sm, sep. DM)

the j-invariant induce a map

$$j : \mathcal{M}_{1,1} \longrightarrow \mathbb{A}^1$$

In general, functors arising in moduli theory are of the form $M : Sch^{op} \longrightarrow Set$

$T \longmapsto$ (iso class of certain geo. obj. $/T$)

can be lifted to

$$\mathcal{M} : Sch^{op} \longrightarrow Gpd$$

$$T \longmapsto (\text{groupoid of the geo. objs}/T)$$

By mod iso we obtain $\mathcal{M} \longrightarrow M$

1.5. (Gerbes) $\quad H^1(\mathcal{C}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Tors}(\mathcal{C}, \mathcal{F})/\text{iso}$

groupoid of $\mathcal{F}$-torsors

what about $H^2(\mathcal{C}, \mathcal{F})$

$$\downarrow \wr$$

$\mathrm{Gerb}(\mathcal{C}, \mathcal{F})/\text{equiv.}$

2-cat of gerbes (a kind of stack)

$X \in \mathcal{C}$, $\mathcal{F}$ sheaf of gp

$$\frac{\mathcal{Y} \quad \mathcal{F} \to h_X}{\mathcal{F} \times_{h_X} \mathcal{Y} \longrightarrow \mathcal{Y}}$$

1.6. (failed to be a sheaf)

G S-gp schm

$\underline{H^1(-, G)} : (Sch/_S)^{op} \longrightarrow Set$

$X \longmapsto \underline{H^1(X, G_X)} = Tors(X, G_X)/iso$

is not a sheaf ($\underline{eg}$. $F \in Ab(Set)$

$\mathcal{H}^n(F) := H^n(-, \underline{F}) : Set^{op} \longrightarrow Ab$.

$\neq 0$

For any $n \geq 1$. $\mathcal{H}^n(F)^+ = 0$)

$\leadsto$ BG : $(Sch/_S)^{op} \longrightarrow Gpd$.

$T \longmapsto Tors(T, G_T)$ ~~/iso~~

is a sheaf in groupoid (i.e. a stack)

§2 Descent & sheaves

<u>2.1</u>. Recall that the objs in $\mathrm{Shv}(\mathcal{C})$ are those (product preserving functors)

$$\mathcal{F} : \mathcal{C}^{op} \longrightarrow \mathrm{Set} \quad \text{s.t.}$$

$$\mathcal{F}(U_{-1}) \xrightarrow{\sim} \lim\left(\mathcal{F}(U_0) \rightrightarrows \mathcal{F}(U_1)\right)$$

for all covering $U_0 \to U_{-1}$

$$\left(\text{with } U_1 = U_0 \times_{U_{-1}} U_0\right).$$

2.2 **Def** (Descent, zheng) Let $f : X_0 \to X_{-1}$ be an *edge* of $\mathcal{C}$ (∞-cat)

$F : \mathcal{C}^{op} \to \mathcal{D}$. (functor of ∞-cat).

2.4 *Čech nerve* of f $X_\bullet : N(\Delta_+)^{op} \to \mathcal{C}$

$[n] \longmapsto \underbrace{X_0 \times_{X_{-1}} \cdots \times_{X_{-1}} X_0}_{n+1}$

f is of *F-descent* if

$F \circ X_\bullet^{op} : N(\Delta_+) \to \mathcal{D}$ is a *limit diagram*.

$[-1] = \emptyset$ $[n] = \{0, 1, \cdots n\}$

i.e. $\quad F(X_{-1}) \xrightarrow{\sim} \lim\limits_{[n]\in\Delta} F(X_n)$

$$\| $$

$$\lim\Big(F(X_0) \rightrightarrows F(X_1) \overset{\rightarrow}{\rightrightarrows} F(X_2) \overset{\rightrightarrows}{\rightrightarrows} \cdots \Big)$$

$$\begin{tikzcd}
\circ \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \circ \arrow[d] \\
\circ \arrow[r] & \circ
\end{tikzcd}$$

$$\begin{tikzcd}
 & F(X_0) & F(X_1) & F(X_2) \\
\underline{F(X_{-1})} \arrow[ur] \arrow[urr] \arrow[urrr] & \xrightarrow{\sim} & & \\
 & & G \arrow[uul] \arrow[uu] \arrow[ull, dashed] &
\end{tikzcd}$$

下降图本

descent datum.

2.3. Def (stacks) Let S be a scheme.

A functor $\mathcal{X} : (Sch_{/S})^{op} \longrightarrow Espd$ is a stack$/S$ if $\mathcal{X}$ preserve products and every fppf mor. f in $Sch_{/S}$ is of

$f : U_0 \to U_{-1}$

$\mathcal{X}$ - desent. (i.e.

$$\mathcal{X}(U_{-1}) \xrightarrow{\sim} \underline{\lim\left(\mathcal{X}(U_0) \rightrightarrows \mathcal{X}(U_1) \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathcal{X}(U_2)\right)}$$

$DD(U_0 \to U_{-1})$

desent datum

2.4 Rmk ① In general, if the ∞-cat $\mathcal{C}$ is equip with Groth top τ,

$F \in \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}) = \mathrm{PSh}(\mathcal{C})$ is

τ-sheaf if

$$\begin{tikzcd} N(\Delta_+) \arrow[r, "\check{C}(f)"] \arrow[dr] & \mathrm{PSh}(\mathcal{C}) \arrow[d, "\mathrm{Map}_{\mathrm{PSh}(\mathcal{C})}(-, F)"] \\ & \mathcal{S} \end{tikzcd}$$

j Yoneda $\coprod j(U_i) \to j(U)$

is a limit diagram for every $\{U_i \to U\}_i$

$$\mathrm{Shv}(\mathcal{C}_\tau) \hookrightarrow \mathrm{PSh}(\mathcal{C})$$

one can also use sieves to describe sheaves.

$$(\mathrm{stacks}/S) \simeq \mathrm{Shv}(N(S_{fppf})) = N(\mathrm{Shv}(S_{fppf}, \mathrm{Gpd}))$$

② (Groth. construction)

one can also use fibered cats to describe stacks.

$$\mathcal{X} : \mathrm{Sch}^{op} \to \mathrm{Gpd} \quad \rightsquigarrow \quad \mathcal{X} \to \mathrm{Sch}$$

fibered cat in groupoid

[HTT] $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Cat}_\infty) \underset{\mathrm{St.}}{\overset{\mathrm{un.}}{\rightleftarrows}} (\mathrm{Cat}_\infty)^{\mathrm{cart}}_{/\mathcal{C}}$

§3. Representability

3.1. Lemma (2-Yoneda)

$$
\begin{aligned}
\mathrm{Sch}_{/S} &\longrightarrow \mathrm{Fun}(\mathrm{Sch}_{/S}^{op}, \underline{\mathrm{Gpd}}) \\
U &\longmapsto h_U
\end{aligned}
$$

(also denoted by U)

be viewed as a stack and $\mathcal{X} \in \mathrm{Fun}(\mathrm{Sch}_{/S}^{op}, \mathrm{Gpd})$. Then natural equ. of gpd $\mathcal{X}(U) \xrightarrow{\sim} \mathrm{Fun}(h_U, \mathcal{X})$.

3.2 Prop All cats in diagram 1.3 admits pullbacks.

3.3 ~~Def~~ A morphism $f: Y \to X$ of

$\mathrm{shv}(S_{fppf}, \mathrm{Set})$ (resp. $\mathrm{shv}(S_{fppf}, \mathrm{Gpd})$)

schemes stacks

is representable if for every $U \in Sch/S$

and $\forall\, x \in X(U)$ (by 3.1, this amounts to give a mor $U \to X$)

$Y \times_X U \in Sch/S$ (resp. $\cancel{Esp}/S$)

$$
\begin{tikzcd}
\text{alg spaces} \arrow[r] & Y \times_X U \arrow[r] \arrow[d] & U \arrow[r] \arrow[d, "x"] & \text{schemes} \\
& Y \arrow[r, "f"] & X & \text{stacks} \arrow[l]
\end{tikzcd}
$$

LZ Six operations on higher Artin stacks and for them . . . .

$$
\begin{tikzcd}
O \arrow[r] \arrow[d] & U \arrow[d] \\
Y \arrow[r] & X
\end{tikzcd}
$$

sm, sm

$$
X \xrightarrow{\text{ét, sm}} X \leftarrow \text{Dm}
$$

$U \xrightarrow{f} V$

$X_V \xrightarrow{f^*} X_U$

$X \to Sch$

$(x, U) \mapsto U$

$x \in X(U)$

[SP, Olsson16 stack, HTT,

fibered cat

$Fun(Sch^{op}, Gpd)$

Aff^{op}

Lishizhong $\boxed{BG}$

cartesian

alg space

Sch_{fppf} $Sch_{ét}$

stack