Local-global principle on alg. stacks

§1. Generalities

1.1. Points Let $\mathcal{X}$ and T be stacks over S.

$\mathcal{X}(T) := \mathrm{Hom}_S(T, \mathcal{X})/_{\cong}$ — T-points of $\mathcal{X}$.

$S = \mathrm{spec}\, k$

In particular $\mathcal{X}(\mathbb{A}_k)$ — adèlic pts

$\mathcal{X}(k)$ — rat. pts.

$\mathcal{X}(\mathcal{O}_k)$ — int. pts.

<u>1.2</u>. Caution ☡ Although if X is rep. by a scheme, this notion coincides with classical one. But

$$\underline{X(k) \longrightarrow X(\mathbb{A}_k)} \text{ is NOT ness. inj.}$$

<u>Eg</u>. Let G be a sm affine k-gp.

$$\ker\left(BG(k) \longrightarrow BG(\mathbb{A}_k)\right) = \ker\left(H^1(k,G) \longrightarrow H^1_{fppf}(\mathbb{A}_k, G)\right)$$

$$= Ш^1(G/_k)$$

1.3 Cohomological obstruction.

$\tau \in \{ \text{ét}, \text{fppf} \}$. $\qquad \mathcal{G} \in \mathrm{Grp}(S_\tau)$

$F = H^i(-, \mathcal{G}) : \mathrm{Sh}(S_\tau, \mathrm{Gpd}) \longrightarrow \mathrm{Set}$.

is a 2-functor (2-iso mapped to identity)

$$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow s"] & \cdot \end{tikzcd} \xrightarrow{F} \begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\|"] & \cdot \end{tikzcd}$$

$\rightsquigarrow X(\mathbb{A}_k)^A \quad (A \in F(X))$

$X(\mathbb{A}_k)^F$ are well def. and

$X(k) \hookrightarrow X(\mathbb{A}_k)^F \subseteq X(\mathbb{A}_k)^A \subseteq X(\mathbb{A}_k)$

In particular, we have $X(\mathbb{A}_k)^{Br}$, $X(\mathbb{A}_k)^{desc}$

§2. A stacky curve violating local-global principle for int. pts.

2.1. Def (stacky curve) . . . .

2.2 Def (Genus) — - . . .

2.3 If $g(X) < \frac{1}{2}$, k - field [BP22]

(A+) local-global principle holds and [Chr20] satisfies strong app. Thus looking for $g(X) = \frac{1}{2}$ [BP22] counter-example for $k = \mathbb{Q}$

2.4. "Thm" (Wu-L 22) k # field

$\exists\,(p,q)$ s.t. stacky curve $\mathcal{X}_{(p,q)}$ (of genus $\frac{1}{2}$) violating local-global principle for int. pts.

$$\mathcal{Y}_{(p,q)} := \mathrm{Proj}\left(\mathcal{O}_K[x,y,z]/(z^2-px^2-qy^2)\right)$$

$\mu_2 \curvearrowright$ $(x:y:z) \longmapsto (x:y:\lambda z)$

$\lambda \in$

Then $\mathcal{X}_{(p,q)} := [\mathcal{Y}_{(p,q)}/\mu_2]$.

§ 3. Descent by gerbes

3.1. • Recall that descent by torsors

$$X(\mathbb{A}_k)^f = \bigcup_{\sigma \in H^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k))$$

for any $[f : Y \to X] \in H^1(X, G)$

• We already know [BRAPS 5.5]

H^2 classifies gerbes.

§ 2 "Png", (L 21) (descent by gerbe)

consider the cat of stacks over k.

$\mathrm{Shv}(k_{fppf}, \mathrm{Gpd})$. $\tau \in \{fppf, \acute{e}t\}$.

For any $\mathcal{G} \in \underline{Ab}(k_\tau)$ and $[f : \mathcal{Y} \to \mathcal{X}]$

$\in H^2_\tau(\mathcal{X}, \mathcal{G})$. we have

$$\mathcal{X}(\mathbb{A}_k)^f = \bigcup_{\sigma \in H^2_e(k, \mathcal{G})} f^\sigma(\mathcal{Y}^\sigma(\mathbb{A}_k))$$

3.3 Def (2-descent ob)

$$X(\mathbb{A}_k)^{2\text{-desc}} := \bigcap_{\substack{G \text{ comm.} \\ \text{linear } k\text{-gp}}} X(\mathbb{A}_k)^{H^2_{et}(-,G)}$$

$\mathbb{G}_m$

by 3.2

$$= \bigcap_{G,\ f : Y \to X \in \mathrm{Cent}(X,G)} \bigcup_{\sigma \in H^2(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k)).$$

3.4 . [Harari 02] $\Rightarrow$ $X(\mathbb{A}_k)^{2\text{-desc}} = X(\mathbb{A}_k)^{Br}$ for var.

The key is to use Poitou–Tate for k.

and $\mathrm{Br}\, X \to$ is torsion

- For $\begin{cases} \text{reg. noe. DM stacks [A. M. 20]} \\ [X/G] \quad X/k \text{ reg.} \quad G \text{ linear } k\text{-gp.} \end{cases}$

$\mathrm{Br}\, X$ is also torsion! "[Wu–L. 22]".

$$\Rightarrow \quad X(\mathbb{A}_k)^{2\text{-desc}} = X(\mathbb{A}_k)^{\mathrm{Br}}.$$

Moreover, $\qquad = X(\mathbb{A}_k)^{\mathrm{conn.}}$

3.5 Construction.

welldef

$$X(\mathbb{A}_k)^{2\text{-desc},\,\text{desc}} = \bigcap_{\substack{G \text{ conn} \\ f : Y \to X \in \mathrm{Gerb}(X,G)}} \ \bigcup_{\sigma \in H^2(k,G)} f^{\sigma}\left(Y^{\sigma}(\mathbb{A}_k)^{\text{desc}}\right)$$

$$\subseteq X(\mathbb{A}_k)^{\text{desc}}.$$

Q.

- counter-example. for $\subsetneq$?
- or proof for $=$?

§4. More on B-M of

4.1 "Thm" (Wu-[22]) (Sansuc exact seq for quotient stacks). Let X/k var. k char $=0$. G conn k-gp. $\curvearrowright X$.

$\underline{Y} = [X/G]$ $\quad U := \mathbb{G}_m/k^\times \in PSh(Y)$ where $k^\times$ is const. Then we have exact seq

$$0 \to UY \to UX \to UG \to \operatorname{Pic} Y \to \operatorname{Pic} X \to \operatorname{Pic} G \to$$
$$\operatorname{Br} Y \to \operatorname{Br} X \to \operatorname{Br}(G \times_k X)$$

4.2 "Cor" In paricitalar for $\mathrm{Br}\,G$,

$U\,BG = 0$, $\mathrm{Pic}\,BG = U\,G$ and

$$0 \to \mathrm{Pic}\,G \to \mathrm{Br}\,BG \to \mathrm{Br}\,k \to 0$$

splits.

4.3 "thm" (Wu-L 22) (Fundamental seq of CT)

$p : X \to k$ alg. stack of f.t., k # field

S k-gp of mul type $\hat{S}$ Cartier dual

$$KD'(\mathcal{X}) := \mathrm{cone}\left(\mathbb{G}_m[1] \longrightarrow Rp_* \mathbb{G}_m[1] \right).$$

in $D^b(k_{\acute{e}t})$.

Then we have the fund. ex seq

$$H^1(k, S) \hookrightarrow H^1_{fppf}(\mathcal{X}, S) \xrightarrow{\chi} \mathrm{Hom}_{D(k)}(\hat{S}, KD'(\mathcal{X}))$$

$$\longrightarrow H^2(k, S) \to H^2_{fppf}(\mathcal{X}, S) \quad \text{where}.$$

χ is the extended type.

<u>4.4</u> Let $\underline{a} \in H^1(k, \hat{S})$ the diag.

$$\begin{tikzcd} H^1(X, S) \arrow[r, "\chi"] \arrow[d, "p^*(a) \cup -"'] & \mathrm{Hom}_{D(k)}(\hat{S}, KD'(X)) \arrow[d, "a \cup - = \lambda_* -"] \\ \mathrm{Br}_1 X \arrow[r, "r"] & H^1(k, KD'(X)) \end{tikzcd}$$

commc.

For $f: Y \xrightarrow{S} X \in \mathrm{Tors}(X, S)$, define.

$\lambda = \chi([f])$ and

$\mathrm{Br}_\lambda X = r^{-1}(\lambda_*(H^1(k, \hat{S}))) \subseteq \mathrm{Br}_1 X$.

4.5. "Prop" We have $X(\mathbb{A}_k)^f = X(\mathbb{A}_k)^{\mathrm{Br}\,\lambda}$

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the big plan: DESCENT for all known ob. (including their relations).