Cohomological descent for obstructions I

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§0. Introduction

k - # field

- We recall that for sm, gp, geo. int k-var,

$$X(\mathbb{A}_k)^{Br} = X(\mathbb{A}_k)^{PGL} = X(\mathbb{A}_k)^{conn} =$$
$$X(\mathbb{A}_k)^{2\text{-desc}} = X(\mathbb{A}_k)^{h\mathbb{Z}} \supseteq X(\mathbb{A}_k)^{h} =$$
$$X(\mathbb{A}_k)^{desc,desc} = X(\mathbb{A}_k)^{fin,desc} = X(\mathbb{A}_k)^{et,Br} =$$
$$X(\mathbb{A}_k)^{desc}$$

([Har02, Foo07, Sko07, Sko09, Dem09, Pa10, HS13, CDX19, Cao20])

- Question: what about on algebraic stacks

§1. Brief intro to algebraic spaces/stacks

Notation $\mathrm{Sch} \subset \mathrm{Esp} \subset \mathrm{Chp}$

(coprod of) qs schemes, alg. spaces, alg stacks

cats are ∞-cats.

Extension of geo. cats.

$(\hat{\mathcal{C}}, \hat{\mathcal{E}})$ marked cat $\hat{\mathcal{E}}$ edges containing degeneration s.t.

- $\hat{\mathcal{E}}$ adm pb
- $\hat{\mathcal{E}}$ stable under composition & pb.

$\mathcal{C} \subseteq \hat{\mathcal{C}}$ full subcat s.t.

- stable under pb
- $\forall X \in \tilde{\mathcal{C}}$, $\exists\ Y \to X$ in $\tilde{\mathcal{E}}$, rep. in $\mathcal{C}$ w/ Y in $\mathcal{C}$.
- $\mathcal{E} = \tilde{\mathcal{E}} \cap \mathcal{C}$

$\rightsquigarrow (\mathcal{C}, \mathcal{E})$ marked.

Eg (1) $\mathcal{C} = \mathrm{Sch}$, $\tilde{\mathcal{C}} = \mathrm{Esp}$.

$\tilde{\mathcal{E}} =$ sm. ét surj

(2) $\mathcal{C} = \mathrm{Esp}$. $\tilde{\mathcal{C}} = \mathrm{Chp}$.

$\tilde{\mathcal{E}} =$ sm. surj

Def. (1) An alg space is a sheaf $X : \mathrm{Sch}_{fppf}^{op} \longrightarrow \mathrm{Set}$ s.t.

① diag $X \to X \times X$ is rep (by sch)

② s.t. extension of Eg (1) i.e. sch $Y \to X$

(2) An alg stack is a "sheaf" $X : \mathrm{Sch}_{fppf}^{op} \longrightarrow$ Groupoid s.t.

① diag $X \to X \times X$ is rep (by Esp)

② s.t. ext. of Eg (2).

(equivalent to let $\underset{\text{sch}}{Y} \to X$)

called "atlas"

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§2 Enhanced six op's.

[LZ17]. Λ a ring. Liu–Zheng construct a S - base sch. Sch $\supset$ functor

$$D : (\mathrm{Chp}_{/S})^{op} \to \mathrm{Pr}^L$$

$$X \longmapsto D(X, \Lambda)$$

$$f : Y \to X \longmapsto f^* : D(X, \Lambda) \to D(Y, \Lambda)$$

1L

s.t. ① $hD(X, \Lambda) = D_{cart}(X_{lis\text{-}et}, \Lambda)$

② $\exists$ right adj $f_* : D(Y, \Lambda) \to D(X, \Lambda)$

1R

Rk ① also constr $f_! : D(Y) \rightleftarrows D(X) : f^!$

2L

for f loc. of f.t., $\Lambda \in \mathrm{Ring}_{\square\text{-}f}$

2

X is ℓ-coprime. ②.

$$3L \quad - \otimes - : D(X, \Lambda) \times D(X, \Lambda) \to D(X, \Lambda)$$

$$3R : \mathcal{H}om(-, -) : D(X, \Lambda)^{op} \times D(X, \Lambda) \to D(X, \Lambda).$$

From now on we omit Λ

Properties: ① (Künneth) Given pb.

$$\begin{array}{ccc} Y & \xrightarrow{(q_1 \ldots q_n)} & Y_1 \times \ldots Y_n \\ f \downarrow & \lrcorner & \downarrow f_1 \times \ldots f_n \\ X & \xrightarrow{(p_1 \ldots p_n)} & X_1 \times \ldots X_n \end{array}$$

$\to$ loc. ~~f.t.~~ + $\square$

(5) (coh. des)

$f : X_0 \to X_{-1}$ sm surj.

$X_\bullet : \Delta_+^{op} \longrightarrow \mathrm{Chp}_{/S}$ Čech nerve

(a) $D(X_{-1}) \xrightarrow{\sim} \varprojlim_{n \in \Delta} D(X_n)$,

$*$-pb on $!$-pb $\otimes$

(b) $D(X_{-1}) \xrightarrow{\sim} \varprojlim_{n \in \Delta} D(X_n)$

§3. Cats encoding obstructions

$$\begin{tikzcd} A \arrow[rr, "x"] \arrow[dr, "q"'] & & X \arrow[dl, "p"] \\ & S & \end{tikzcd}$$

- Basicly we shall use

$\mathcal{H}(X) := \mathrm{Fun}(D(A), D(X))$

$\rightsquigarrow \mathcal{H} : (\mathrm{Chp}_{/S})^{op} \longrightarrow \mathrm{Pr}^L$

$f : Y \to X \longmapsto f^* : \mathrm{Fun}(D(A), D(X))$

$\to \mathrm{Fun}(D(A), D(Y))$

cohdes of $f : X_0 \to X_{-1}$ $\rightsquigarrow$ $\mathcal{H}(X_{-1}) \xrightarrow{\sim} \varprojlim_{n \in \Delta} \mathcal{H}(X_n)$

- However, this is too large!

want to control ~~cohomological pul~~ pullback $H^n(-, F)$

Observe that for $x \in X(A)$,

$p_* x_* \xrightarrow{\sim} q_*$ and $x \in X(A)^{Br} \Leftrightarrow$ (5)

$p^* : H^2_{et}(X, \mathbb{G}_m) \to H^2_{et}(S, \mathbb{G}_m)$ comes from q^*

Def. Define ccits $\mathcal{H}_0, \mathcal{H}_1$ by the ~~p~~ cart. square.

$$\begin{tikzcd}
\mathcal{H}_1(X) \arrow[r, "v"] \arrow[d, "Q_*"] & \mathcal{H}_0(X) \arrow[r, "u"] \arrow[d, "P_*"] & \mathcal{H}(X) \arrow[d, "P_*"] \\
\mathcal{H}(S)_{q_*//q_*} \arrow[r, "v"] & \mathcal{H}(S)_{q_*/} \arrow[r, "u"] & \mathcal{H}(S)
\end{tikzcd}$$

~~Given x_*, λ_*~~

$$\begin{tikzcd}
A \arrow[rr, "x_*"] \arrow[dr, "q_*"'] & \arrow[d, phantom, "\Rightarrow\lambda_* \ \Leftarrow\mu_*"] & X \arrow[dl, "p_*"] \\
& S &
\end{tikzcd}
\qquad
\begin{tikzcd}
q_* \arrow[d, "\lambda_*"] \arrow[dd, bend right, "id"'] \\
p_*x_* \arrow[d, "\mu_*"] \\
q_*
\end{tikzcd}$$

$q_* \xrightarrow{\lambda_*} p_*x_*$

$$\begin{tikzcd}
(x_*, \lambda_*, \mu_*) \in \mathcal{H}_1(X) \arrow[d, mapsto, "v"] \\
(x_*, \lambda_*) \in \mathcal{H}_0(X) \arrow[d, mapsto, "u"] \\
x_* \in \mathcal{H}(X)
\end{tikzcd}$$

- controlling H'' ~~E~~ pullback

- Suppose ~~...~~ $\bar{x}_* = (x_*, \lambda_*, \mu_*) \in \mathcal{H}_1(X)$

s.t. x_* has a left adjoint $x^*: D(X) \to D(A)$

$\leadsto \quad q^* \xrightarrow{\mu^*} x^*p^* \xrightarrow{\lambda^*} q^*$

Let $K_1, K_2 \in D(S)$. & $\alpha: p^*K_1 \to p^*K_2$

be given. $\leadsto$

$$q^*K_1 \xrightarrow{\mu^*K_1} x^*p^*K_1 \xrightarrow{x^*\alpha} x^*p^*K_2 \xrightarrow{\lambda^*K_2} q^*K_2 .$$

edge in $D(A)$

$$\begin{tikzcd}
A & & X \arrow[ll, "x^*"'] \\
& S \arrow[ul, "q^*"] \arrow[ur, "p^*"'] \arrow[u, phantom, "\Uparrow\mu^* \ \Downarrow\lambda^*"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\alpha: p^*K_1 \arrow[d] \\
p^*K_2
\end{tikzcd}$$

$\leadsto_{\bar{x}^*_1} \quad \mathrm{Hom}_{D(X)}(p^*K_1, p^*K_2) \to \mathrm{Hom}_{D(A)}(q^*K_1, q^*K_2)$

homomorphism

Given $\bar{x}_{1*} \to \bar{x}_{2*}$ in $\mathcal{H}_1(X)$ hence

$$\begin{tikzcd}
q^*K_1 \arrow[r] \arrow[d, equal] & x_2^*p^*K_1 \arrow[r] \arrow[d] & x_2^*p^*K_2 \arrow[r] \arrow[d] & q^*K_2 \arrow[d, equal] \\
q^*K_1 \arrow[r] & x_1^*p^*K_1 \arrow[r] & x_1^*p^*K_2 \arrow[r] & q^*K_2
\end{tikzcd}$$

upshot $\quad h\bar{x}_1^* = h\bar{x}_2^*$

In particular $\Rightarrow x_1^* = x_2^* : H^1_{et}(\cancel{X_K}, K) \to H^1_{et}(A, K)$

$K_1 = A,\ K_2 = K[i]$ torsion

Def $X(A)^{obs} \subseteq X(A)$ subset given.

(1) Define $\mathcal{A}_1^{obs}(X) \subseteq \mathcal{H}_1(X)$ full sub cat spanned by $\bar{x} = (x_*, \mathcal{E}, \mathcal{E}^{-1}) \in \mathcal{H}_1(X)_0$, $x \in X(A)^{obs}$, $\mathcal{E}: \eta_x \xrightarrow{\sim} x_* p_*$.

$\mathcal{B}_1^{obs}(X) \subseteq \mathcal{H}_1(X)$ — — — — — — — — — s.t. $\exists$ edge $\bar{x}_* \to \bar{x}_{0*}$ of $\mathcal{H}_1(X)$ where $\bar{x}_{0*} \in \mathcal{A}_1^{obs}(X)_0$.

(2) $\mathcal{A}_0^{obs}(X) \subseteq \mathcal{H}_0(X)$ full sub span by ess im of $\mathcal{A}_1^{obs}(X)$ under $V: \mathcal{H}_1(X) \to \mathcal{H}_0(X)$

$\mathcal{B}_0^{obs}(X) \subseteq$ — — — — — — — — — $\mathcal{B}_0^{obs}(X)$ — — — V.

Rk (a) in defs of $\mathcal{A}_0^{obs}$, $\mathcal{B}_0^{obs}$, ess im can be just im

(b) Let $\bar{x}_* \to \bar{x}_{0*}$ edge of $\mathcal{H}_0(X)$ where $\bar{x}_{0*}$ is ess im of $\mathcal{A}_1^{obs}(X)$ under V.
$\Rightarrow \bar{x}_* \in \mathcal{B}_0^{obs}(X)_0$.

§4 Cohomological descent for obstruction

$f: Y \to X$ map in $\mathrm{Chp}_{/S}$

Lemma. (adj) (1) $f^*: \mathcal{H}(X) \rightleftarrows \mathcal{H}(Y) : f_*$

induces $\overset{\text{adj}}{}$ $f^*: \mathcal{H}_0(X) \rightleftarrows \mathcal{H}_0(Y) : f_*$

(2) If $f: Y(A)^{obs} \to X(A)^{obs}$

is surj, then

$f^*: \mathcal{H}_0(X) \rightleftarrows \mathcal{H}_0(Y) : f_*$ induces

adj $f^*: B_0^{obs}(X) \rightleftarrows B_0^{obs}(Y) : f_*$.

Pf. By def & checking. □

Lemma (rad) $f: X_0 \to X_{-1}$ sm in $\mathrm{Chp}_{/S}$.

$\cap$ $\mathrm{Exp}_{/S}$. $X_\bullet: \Delta_+^{op} \to \mathrm{Chp}_{/S}$ Čech.

then $\forall\ \alpha: [m] \to [n]$ in Δ_+,

$$\begin{array}{ccc} \mathcal{H}_0(X_m) & \xrightarrow{d_0^*} & \mathcal{H}_0(X_{m+1}) \\ \downarrow \alpha^* & & \downarrow \\ \mathcal{H}_0(X_n) & \xrightarrow{d_0^*} & \mathcal{H}_0(X_{n+1}) \end{array}$$

is right adjointable.

Pf. For D, this is known.

$\cancel{RD}$ $\cancel{\mathcal{H}_0}$ $\overset{\text{Lemma (adj) (1)}}{\rightsquigarrow}$ also for $\mathcal{H}_0$ □

• Lemma (rad)

• Poincaré $\Rightarrow$ $f^* : \mathcal{H}o(X_{-1}) \to \mathcal{H}o(X_0)$
□ condition
preserves small lim

• f sm surj $\Rightarrow$ f^* conservative

$\Rightarrow$ Lemma $f : X_0 \xrightarrow{sm,\, surj} X_{-1}$ in $Chp_{/S}$. $\Lambda \in$ Ring ...

$X_0 \in Esp_{/S}$

Then ⓒ $\mathcal{H}o(X_{-1}) \xrightarrow{\sim} \varprojlim_{n \in \Delta_s} \mathcal{H}o(X_n)$.

Restricting to B_0^{obs} +

$\underbrace{\text{lifting property of } B_0^{obs}}$

$$\left(\begin{array}{l} \bar{x}_* \to \bar{x}_{0*} \text{ in } \mathcal{H}o(X) \\ \quad\quad\quad \text{ess of } A_1^{obs}(X) \\ \Rightarrow \bar{x}_0 \in B_0^{obs} \end{array} \right)$$

$\Rightarrow$ Lemma $f : X_0 \to X_{-1}$ sm surj in $(Chp_{/S}^{\mathcal{F}^{obs},\square})_{\mathcal{F}^{obs}}$.

$X_0 \in Esp_{/S}^{\mathcal{F}^{obs},\square}$

Then $B_0^{obs}(X_{-1}) \xrightarrow{\sim} \varprojlim_{n \in \Delta_s} B_0^{obs}(X_n)$

Notation
• $\mathcal{F}_1, \mathcal{F}_2$ two family of maps of $Chp_{/S}$.

$(Esp_{/S}^{\mathcal{F}_1})_{\mathcal{F}_2} \subseteq Esp_{/S}$ subcat sp by
- obj. adm. $\mathcal{F}_1 \cap (Esp_{/S})_0$ - atlas
- mor in $\mathcal{F}_2 \cap (Esp_{/S})_1$

Similarly for $(Chp_{/S}^{\mathcal{F}_1})_{\mathcal{F}_2}$

$\mathcal{F}^{obs}$ ...

Extend $\mathcal{C}^{obs}$ by

$$\mathrm{Fun}^{\Sigma}((\mathrm{Chp}^{\Sigma}_{/S})^{op}_{rep}, \mathrm{Cat}_\infty) \xrightarrow{\sim}$$

$$\cdots (\mathrm{Esp}^{\Sigma}_{/S})^{op}_{rep} \cdots \xrightarrow{\sim}$$

$$\cdots (\mathrm{Sch}_{/S})^{op}_{rep} \cdots$$

$$\rightsquigarrow \mathcal{C}^{obs} \in \mathrm{Fun}^{RAd, \Sigma}((\mathrm{Chp}^{\Sigma}_{/S})^{op}_{rep}, \mathrm{Cat}_\infty) / \mathrm{Pt}_0$$

~~s.t. $\forall f : X_0 \to X_{-1} \in$ Ch~~

Thm $(\mathcal{C}) \wedge \ldots$ ~~$\forall X \in \mathrm{Chp}$ given~~

$\forall X \in \mathrm{Sch}_{/S}$, given $X(A)^{obs} \subseteq X(A)$

Σ s.t. (a) (b) (c) (d).

i.e. $obs : \mathrm{Sch}_{/S} \to \mathrm{Set}$

$$X \longmapsto X(A)^{obs}.$$

Then $\exists$ way to $\forall (obs, \Sigma)$, a fun

$$\mathcal{C}^{obs}_{\Sigma} : (\mathrm{Chp}^{\Sigma}_{/S})^{op}_{rep} \longrightarrow \mathrm{Cat}_\infty \quad \text{s.t.}$$

$\mathcal{C}^{obs}_{\Sigma}$ is smallest

(1) $\forall X \in \mathrm{Sch}$, $A^{obs}_0(X) \subseteq \mathcal{C}^{obs}_{\Sigma}(X)$.

(2) descent $\mathcal{C}^{obs}_{\Sigma}(X_{-1}) \xrightarrow{\sim} \cdots$

(3) $\mathcal{C}^{obs}_{\Sigma} \subseteq \mathcal{H}_0$ compatible

(4) $obs_1 \subseteq obs_2 \Rightarrow \mathcal{C}^{obs_1}_{\Sigma} \subseteq \mathcal{C}^{obs_2}_{\Sigma}$

(5) if $obs_{ext} : \mathrm{Chp}^{\Sigma}_{/S} \to \mathrm{Set}$

$$X \longmapsto X(A)^{obs} \subseteq X(A)$$

is functorial, and

$\forall\ f: Y \to X$ in $\mathcal{E} \cap \mathrm{Sch}$,

$Y(A)^{obs} \twoheadrightarrow X(A)^{obs}$. then $\mathcal{C}_{\mathcal{E}}^{obs}(X \subseteq \mathrm{Bo}(X$

$\forall X \in \mathrm{Chp}^{\mathcal{E}}_{/S}$

(6) $\mathcal{E}_0 \subset \mathcal{E} \Rightarrow \mathcal{C}^{obs}_{\mathcal{E}_0} \subseteq \mathrm{res}_c\, \mathcal{C}^{obs}_{\mathcal{E}}$

§5. New obstruction by $\mathcal{C}$.

Def. (1) $X(\mathbb{A}_k)^{\widetilde{obs_{\mathcal{E}}}} := X(k) \cup \{x \in X(\mathbb{A}_k) \mid$

$\bar{x}_* = (x_*, \mathcal{E}) \in \mathrm{Ho}(X)_\epsilon$

lies in ~~im~~ $\mathcal{C}^{obs}_{\mathcal{E}}(X$

$\}$

(2) $F: \mathrm{Chp}_{/k} \to \mathrm{Set}$ is

functorial if

$F \hookrightarrow \prod_i H^{u_i}_{et}(-, K_i)$

may infinite

~~Ex Eq~~. K_i ~~ob~~ $=$ comm ~~k~~ gp

G_i 17-tor

Fact. ~~B~~ obs functorial, $\Rightarrow$ then

$$\left(\begin{array}{l} x \in X(A) \text{ s.t. } \bar{x}_* \in \mathrm{Bo}^{obs}(X)_\circ \\ \Rightarrow x \in X(A)^{obs} \end{array} \right)$$

Thm (~~o~~ $\widetilde{\text{obs}}$) $\quad Sch'_{/k} \subset Esp'_{/k} \subset Chp'_{/k}$.

Σ (a) (b) (c) (d).

Let $\text{obs} : Sch'_{/k} \to Set$

$X \mapsto X(\mathbb{A}_k)^{\text{obs}}$.

$\forall (\text{obs}, \Sigma) \rightsquigarrow X(\mathbb{A})^{\widetilde{\text{obs}}_\Sigma}$. (~~def~~ by $e^{\text{obs}}_\Sigma(X)$)

an obstruction on $(Chp'_{/k})^\Sigma$ s.t.

(1) $\forall X \in Sch'_{/k}$, $X(\mathbb{A}_k)^{\text{obs}} \subseteq X(\mathbb{A}_k)^{\widetilde{\text{obs}}_\Sigma}$

(2) $\widetilde{\text{obs}}_\Sigma$ is functorial on $(Chp^\Sigma_{/k})^{op}$ resp

(3) $\text{obs}_1 \subseteq \text{obs}_2 \Rightarrow X(\mathbb{A}_k)^{\widetilde{\text{obs}}_{1\Sigma}} \subseteq_\Sigma 2$

(4) $\forall f : Y \to X$ in $\Sigma \cap Sch$, $Y(\mathbb{A})^{\text{obs}} \to X(\mathbb{A})^{\text{obs}}$ ~~$\not\!\!\!$~~. then

(i) ~~$\forall X$~~ obs only for $X \in Sch'_{/k}$,

$\Rightarrow$ ~~o~~ $\widetilde{\text{obs}}_\Sigma$ only for $Chp'^{\,\Sigma}_{/k}$,

(iii) (~~a~~ If + obs ~~c~~ coh. $Chp'^{op}_{/k} \to Set$,

$\Rightarrow \forall X \in (Chp'_{/k})^\Sigma$, $X(\mathbb{A}_k)^{\widetilde{\text{obs}}_\Sigma} \subseteq X(\mathbb{A}_k)^{\text{obs}}$

Impare, on Sch, $=$

(ii) ~~condition~~

$\text{obs} : (Chp'_{/k})^\Sigma \to Set$ ~~and~~ $f : Y \to X$ s.t. ~~?~~

$x \in X(\mathbb{A}_k)^{\widetilde{\text{obs}}_\Sigma}$

s.t. $fy = x$. $\Rightarrow$ sch. $\exists\, y \in$ ~~X~~ $(\mathbb{A}_k)$ $\Rightarrow y \in X(\mathbb{A}_k)^{\widetilde{\text{obs}}_\Sigma}$

(6) $\quad \varepsilon_0 \subseteq \varepsilon \quad \Rightarrow \quad X(\mathbb{A}_k)^{\widetilde{obs_{\varepsilon_0}}} \subseteq \ldots \varepsilon$

(7) $\quad$ If $obs : Ch_{/k}^{\varepsilon} \to Set$ s.t.

$\forall f : Y \to X$ in ε,

$Y(A)^{obs} \twoheadrightarrow X(A)^{obs}$, then

$\forall X \in Ch$, $\quad X(\mathbb{A}_k)^{obs} \subseteq X(\mathbb{A}_k)^{\widetilde{obs_\varepsilon}}$

If + $\overset{obs}{coh}$. $\Rightarrow$ $\quad =$

(8). $\quad C_\varepsilon^{obs} \Rightarrow C_\varepsilon^{\widetilde{obs_\varepsilon}}$ in pari.

$X(\mathbb{A}_k)^{(\widetilde{obs_\varepsilon})_\varepsilon} = \ldots^{\widetilde{obs_\varepsilon}}$

$$\begin{tikzcd}
 & {}_0^1 X \arrow[r, "{}_1^0 f"] & {}_1^0 X \\
 {}_2^0 X \arrow[ru, "{}_1^0 id"] & & \\
 {}_2 X \arrow[u, "{}_2 f"] & &
\end{tikzcd}$$

(19)