Quadratic diophantine equations over function fields

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Outline

1. **Problem:** is $ax^2 + bxy + cy^2 + g = 0$ solvable over $\mathbb{F}_q[t]$?

2. Solution: torsors under norm-one tori + class field theory

3. Application: primes of the form $x^2 + Dy^2$ in $\mathbb{F}_q[t]$

Binary quad. forms over $\mathbb{F}_q[t]$

- Let $k = \mathbb{F}_q$, $F = k(t)$ rational func. field, $\mathfrak{o}_F = k[t]$ the poly. ring.
- Fix a, b, c and d in $\mathfrak{o}_F$, and consider the diophantine eq. def. by the bin. quad. form

$$ax^2 + bxy + cy^2 + g = 0.$$

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An example

Example 1.1

- Consider the eq. over $k[t] = \mathbb{F}_3[t]$

$$-x^2 + txy - (t^3 - t^2 + 1)y^2 + g = 0. \qquad (1.2)$$

Write $g = u \times (t-1)^{s_1} \times (t^2 - t - 1)^{s_2} \times \prod_{j=1}^{r} p_j^{m_j}$, where $u \in k^\times$, p_r distinct monic irr. poly. in $k[t]$.

- Then (1.2) is solvable over $k[t]$ if and only if

(1) $\left(\frac{g \times p^{-v_p(g)}}{p}\right) = (-1)^{v_p(g)}$, for $p = t-1$ or $t^2 - t - 1$,
(2) $\left(\frac{-(t-1)(t^2-t-1)}{p}\right) = 1$, for $p \nmid (t-1)(t^2-t-1)$, $v_p(g)$ odd.

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Example 1.3

- The eq. over $k[t] = \mathbb{F}_3[t]$

$$(t-1)x^2 + (t^2+t-1)y^2 + g = 0. \tag{1.4}$$

Write $q_1 = t-1, q_2 = t^2+t-1$ and $-d = -q_1q_2$ and define

$$\theta(X) = X^4 - (t^2-t)X^2 - t^3 + 1 \in k[t][X]$$

(generates certain Galois extension),

$D_1 = \{p \mid \left(\frac{-d}{p}\right) = 1, \theta(X) \mod p \text{ factors into two irr. polys.}\}$,
$D_2 = \{p \mid \left(\frac{-d}{p}\right) = 1, \theta(X) \mod p \text{ is irr.}\}$
(reflex the splittings of places)

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Another example (count.)

Example (1.3, count.)

- Then (1.4) is solvable over $k[t]$ if and only if

  (1) $\left(\frac{g \times p^{-v_p(g)}}{p}\right) = (-1)^{\deg(p)}$, for q_1 or q_2,
  (2) $\left(\frac{-d}{p}\right) = 1$, for $p \nmid d$ with odd $v_p(g)$,
  (3) either
    - $D_2 = \emptyset$ and $\sum_{p \in \{q_1, q_2\} \cup D_1} v_p(g) \equiv 1 \pmod 2$, or
    - $D_2 \neq \emptyset$ and $\sum_{p \in D_2} v_p(g) \equiv 0 \pmod 2$.

- The conditions (1) and (2) are local,
- and (3) is called the Artin condition.

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The reformulation in arith. geometry

- Restrictions:
  - k with char. not 2, $-d = (b/2)^2 - ac$ not sqare,
  - the quad. ext. $E = F(\sqrt{-d})/F$ is imaginary, i.e., there is a unique place lying over $1/t$.
- The question is to ask the existence of integral points of the affine scheme $\mathbf{X} = \mathrm{Spec}(\mathfrak{o}_F[x,y]/(a(ax^2+bxy+cy^2+g)))$ over $\mathfrak{o}_F$.
  - General tool: (int.) obstructions to local-global principal.
  - Observation: $\tilde{x} = ax + \frac{b}{2}y, \tilde{y} = y, n = -ag$, $\mathbf{X} : \tilde{x}^2 + d\tilde{y}^2 = n$, whose generic fiber admits the structure of a torsor of a torus.
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Torsors under norm-one tori

- Let $T = \ker(R_{E/F}(\mathbb{G}_{m,E}) \xrightarrow{N_{E/F}} \mathbb{G}_{m,F})$ (called norm-one torus), and $X_F = X \times_{\mathfrak{o}_F} F$ the generic fiber.

- Intuition: if $N(t) = 1$ and $N(x) = n$ then $N(tx) = n$.
  $\rightsquigarrow T \times X_F \to X_F$, making X_F a T-torsor.

- Fix a rat. pt. $P \in X_F(F)$ to obtain $\phi_P : X_F \xrightarrow{\sim} T$, whose restriction to $\mathbf{X}(\mathfrak{o}_{F_\mathfrak{p}})$ (int. local pts.) has image in $\mathbb{I}_E$ (idéle group).

- The composition $\tilde{f}_E : \prod_\mathfrak{p} \mathbf{X}(\mathfrak{o}_{F_\mathfrak{p}}) \to \mathbb{I}_E \xrightarrow{\times P} \mathbb{I}_E$ whose $\mathfrak{p}$-component is

$$\tilde{f}_E[(x_\mathfrak{p}, y_\mathfrak{p})] = \begin{cases} (\tilde{x}_\mathfrak{p} + \sqrt{-d}\tilde{y}_\mathfrak{p}, \tilde{x}_\mathfrak{p} - \sqrt{-d}\tilde{y}_\mathfrak{p}) \in E_\mathfrak{P} \times E_{\bar{\mathfrak{P}}} & \text{, if } \mathfrak{p} = \mathfrak{P}\bar{\mathfrak{P}} \text{ splits in } E/F, \\ \tilde{x}_\mathfrak{p} + \sqrt{-d}\tilde{y}_\mathfrak{p} \in E_\mathfrak{P} & \text{, otherwise,} \end{cases}$$

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The sign func. and the positive open subgroup

- Let $L = \mathfrak{o}_F + \mathfrak{o}_F\sqrt{-d} \subseteq \mathfrak{o}_E$ an order, with completion $L_\mathfrak{p} = L \otimes_{\mathfrak{o}_F} \mathfrak{o}_{F_\mathfrak{p}}$ at $\mathfrak{p}$.
- Let $\mathfrak{p}_\infty = 1/t$ be the place of $k(t)$ at ∞ and $\mathfrak{P}_\infty$ the unique place in E above $\mathfrak{p}_\infty$.
- Fix a sign function $\mathrm{sgn} : E_{\mathfrak{P}_\infty} \to \mathfrak{o}_{E_{\mathfrak{P}_\infty}}/\mathfrak{P}_\infty$ (taking the first non-zero coeff. of the Laurent expansion), and take all the positive elements $E^+_{\mathfrak{P}_\infty} = \{\alpha \in E^\times_{\mathfrak{P}_\infty} \mid \mathrm{sgn}(\alpha) = 1\} \subseteq E^\times_{\mathfrak{P}_\infty}$.
- $\rightsquigarrow$ an open subgp. $\Xi^+_{\mathfrak{P}_\infty} = E^+_{\mathfrak{P}_\infty} \times \prod_{\mathfrak{p}\neq\mathfrak{p}_\infty} L^\times_\mathfrak{p} \subseteq \mathbb{I}_E$.

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Existence of int. pts. of **X** (main result)

Theorem 2.1 ([Lv, 2020], func. field analog to [Lv et al., 2018])

Notations above, we have

(a) The open subgp. $E^\times \Xi^+_{\mathfrak{P}_\infty}$ is of finite index in $\mathbb{I}_E$.

(b) Let $K^+_{\mathfrak{P}_\infty}$ be the class field corresp. to $E^\times \Xi^+_{\mathfrak{P}_\infty}$ (called the narrow ring class field) and $\psi_{K^+_{\mathfrak{P}_\infty}/E} : \mathbb{I}_E \to \mathrm{Gal}(K^+_{\mathfrak{P}_\infty}/E)$ the Artin map. Then $\mathbf{X}(\mathfrak{o}_F) \neq \emptyset$ if and only if

- there exists a local solution $\prod_\mathfrak{p}(x_\mathfrak{p}, y_\mathfrak{p}) \in \prod_\mathfrak{p} \mathbf{X}(\mathfrak{o}_{F_\mathfrak{p}})$ (local condition) s.t.
- $\psi_{K^+_{\mathfrak{P}_\infty}/E}(\tilde{f}_E(\prod_\mathfrak{p}(x_\mathfrak{p}, y_\mathfrak{p}))) = 1$ (Artin condition).

- Calculate to obtain Examples 1.1 and 1.3.

Existence of int. pts. of $\mathbf{X}$ (main result)

Theorem 2.1 ([Lv, 2020], func. field analog to [Lv et al., 2018])

Notations above, we have

(a) The open subgp. $E^\times \Xi^+_{\mathfrak{P}_\infty}$ is of finite index in $\mathbb{I}_E$.

(b) Let $K^+_{\mathfrak{P}_\infty}$ be the class field corresp. to $E^\times \Xi^+_{\mathfrak{P}_\infty}$ (called the narrow ring class field) and $\psi_{K^+_{\mathfrak{P}_\infty}/E} : \mathbb{I}_E \to \mathrm{Gal}(K^+_{\mathfrak{P}_\infty}/E)$ the Artin map. Then $\mathbf{X}(\mathfrak{o}_F) \neq \emptyset$ if and only if

- there exists a local solution $\prod_{\mathfrak{p}}(x_{\mathfrak{p}}, y_{\mathfrak{p}}) \in \prod_{\mathfrak{p}} \mathbf{X}(\mathfrak{o}_{F_{\mathfrak{p}}})$ (local condition) s.t.
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- Calculate to obtain Examples 1.1 and 1.3.

Ideas of the proof

- Replace L by $\mathfrak{o}_E \rightsquigarrow \Xi^+_{\mathfrak{P}_\infty}$ by $\tilde{\Xi}^+_{\mathfrak{P}_\infty} = E^+_{\mathfrak{P}_\infty} \times \prod_{\mathfrak{P} \neq \mathfrak{P}_\infty} \mathfrak{o}^\times_{E_{\mathfrak{P}}}$
$\Rightarrow E^\times \tilde{\Xi}^+_{\mathfrak{P}_\infty} / E^\times \Xi^+_{\mathfrak{P}_\infty}$ is finite.
- But $\mathbb{I}_E / E^\times \tilde{\Xi}^+_{\mathfrak{P}_\infty} \cong Cl^+(\mathfrak{o}_E)$ is the narrow class gp., which is finite [Goss, 1996] $\Rightarrow \mathbb{I}_E / E^\times \Xi^+_{\mathfrak{P}_\infty}$ is finite (shows (a)).
- Replace $E^+_{\mathfrak{P}_\infty}$ by $E^\times_{\mathfrak{P}_\infty} \rightsquigarrow \tilde{\Xi}^+_{\mathfrak{P}_\infty}$ by $\Xi_{\mathfrak{P}_\infty} = E^\times_{\mathfrak{P}_\infty} \times \prod_{\mathfrak{P} \neq \mathfrak{P}_\infty} \mathfrak{o}^\times_{E_{\mathfrak{P}}}$.
Call the corresp. $K_{\mathfrak{P}_\infty}$ Hilbert class field.
- The int. model $\mathbf{T} = \mathrm{Spec}(\mathfrak{o}_F[x, y]/(x^2 + dy^2 - 1))$ of T is so that $\mathbf{T}(\mathfrak{o}_{F_\mathfrak{p}})$ acts stably on $\mathbf{X}(\mathfrak{o}_{F_\mathfrak{p}})$ for all $\mathfrak{p}$.
- Use [Wei and Xu, 2012] and exploit the properties of sgn and the def. of $E^+_{\mathfrak{P}_\infty}$ (shows (b)).

Ideas of the proof

- Replace L by $\mathfrak{o}_E \rightsquigarrow \Xi^+_{\mathfrak{P}_\infty}$ by $\tilde{\Xi}^+_{\mathfrak{P}_\infty} = E^+_{\mathfrak{P}_\infty} \times \prod_{\mathfrak{P} \neq \mathfrak{P}_\infty} \mathfrak{o}^\times_{E_\mathfrak{P}}$ $\Rightarrow E^\times \tilde{\Xi}^+_{\mathfrak{P}_\infty} / E^\times \Xi^+_{\mathfrak{P}_\infty}$ is finite.
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Outline

1. Problem: is $ax^2 + bxy + cy^2 + g = 0$ solvable over $\mathbb{F}_q[t]$?

2. Solution: torsors under norm-one tori + class field theory

3. Application: primes of the form $x^2 + Dy^2$ in $\mathbb{F}_q[t]$

Primes of the form $x^2 + Dy^2$ in $k[t]$

- Let $D \in k[t]$ be sqr. free with positive deg., $l \nmid D$ irr., and consider $l = x^2 + Dy^2$ over $k[t]$.
- Suppose that deg D is odd or $\mathrm{lc}(-D) \notin k^{\times 2}$, which is to say $E = F(\sqrt{-D})/F$ is imaginary [Rosen, 2013].
- Then a necessary cond. for int. solvability is

  deg l is even if deg D is,

  which we will always assume.
- Let d_∞ be the relative deg. of $\mathfrak{P}_\infty \mid \mathfrak{p}_\infty$, $\deg^* l = \frac{\deg l}{d_\infty} \in \mathbb{Z}_{>0}$ and choose sgn w.r.t the uniformizer $t^g/\sqrt{-D}$ (g the genus of E).

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Extending a result of Maciak

Corollary ([Lv, 2020], extends [Maciak, 2011])

Notations and assumptions above, we have

(a) if $\mathrm{sgn}(l)(-1)^{\deg^* l} \in k^{\times 2}$, then $l = x^2 + Dy^2$ is solvable over $k[t]$ if and only if

- $\left(\frac{l}{r}\right) = 1$ for each monic irr. factor $r \mid D$ and
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Outline revisited

Thanks for your attention.

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