Obstructions to local-global principle

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§1. Classical coh. obs.

1.1 Obs defined by functors. Let $F : (\mathrm{Sch}/k)^{o} \to \mathrm{Set}$

$x \in X(T)$, $T \xrightarrow{x} X \rightsquigarrow F(X) \xrightarrow{F(x)} F(T)$, $A \longmapsto A(x)$. Then we have an obvious comm. diag

$$\begin{array}{ccc} X(k) & \longrightarrow & X(\mathbb{A}_k) \\ \downarrow A(-) & & \downarrow A(-) \\ F(k) & \longrightarrow & F(\mathbb{A}_k) \end{array}$$

Define $X(\mathbb{A}_k)^A = \{x \in X(\mathbb{A}_k) \mid A(x) \in \mathrm{im}(F(k) \to F(\mathbb{A}_k))\}$, and

$X(\mathbb{A}_k)^F = X(\mathbb{A}_k)^{F(X)} = \bigcap_{A \in F(X)} X(\mathbb{A}_k)^A$

$\Rightarrow X(k) \subseteq X(\mathbb{A}_k)^F \subseteq X(\mathbb{A}_k)^A \subseteq X(\mathbb{A}_k)$

1.2 coh. obs.

- $F = \mathrm{Br} = H^2_{ét}(-, \mathbb{G}_m) \rightsquigarrow X(\mathbb{A}_k)^{\mathrm{Br}}$ B-M ob
- $F = \check{H}^1_{fppf}(-, G)$, G an affine k-gp. $\rightsquigarrow X(\mathbb{A}_k)^{\mathrm{cdesc}} = \bigcap_{\text{all } G} X(\mathbb{A}_k)^{\check{H}^1_{fppf}(X, G)}$

1.3 Descent by torsors (classical)

The (classical) descent of

$\check{H}^1_{fppf}(X, G) \hookrightarrow \mathrm{Tors}(X, G)/\cong$

$[Y] \longmapsto \{f : Y \xrightarrow{G} X\}$.

twisted by $\sigma \in \check{H}^1_{fppf}(k, G)$, $f^\sigma : Y^\sigma \xrightarrow{G^\sigma} X \Rightarrow$

$X(\mathbb{A}_k)^{[Y]} = X(\mathbb{A}_k)^f = \bigcup_{\sigma \in H^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k))$

$\Rightarrow X(\mathbb{A}_k)^{\mathrm{cdesc}} = \bigcap_{G,\ f : Y \xrightarrow{G} X} \ \bigcup_{\sigma \in H^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k))$

1.4 Composite obs

$X(\mathbb{A}_k)^{\mathrm{cdesc},\mathrm{cdesc}} = \bigcap_{G,\ f : Y \xrightarrow{G} X} \bigcup_{\sigma} f^\sigma(Y^\sigma(\mathbb{A}_k)^{\mathrm{cdesc}})$

$X(\mathbb{A}_k)^{ét,\mathrm{Br}} = \text{— — — — — — — — — } \mathrm{Br}$.

- X reg. qp. k-var, $X(\mathbb{A}_k)^{\mathrm{cdesc}} \subseteq X(\mathbb{A}_k)^{\mathrm{Br}}$
- X sm. qp. geo int k-var, $X(\mathbb{A}_k)^{\mathrm{cdesc}} = X(\mathbb{A}_k)^{ét,\mathrm{Br}}$ and $X(\mathbb{A}_k)^{\mathrm{cdesc},\mathrm{cdesc}} = X(\mathbb{A}_k)^{\mathrm{cdesc}}$ (Harari, Skorobogatov … 02~05 | Stoll, Sk, Demarche, Poonen, Xu, Cao, up to 2020)

Question. Can we construct obs smaller than cdesc?

Ideas. Interpolating H^2 by some geo obs, imitate 1.3

§2. Torsors and gerbes.

2.1. ~~Def~~. Let $\mathcal{C}$ a site,

- Let $\mathcal{U} \in \mathrm{Sh}(\mathcal{C})$. A $\mathcal{U}$-gp is a gp obj in $\mathrm{Sh}(\mathcal{C})/\mathcal{U}$
- Any $\mathcal{Y} \in \mathrm{Sh}(\mathcal{C})/\mathcal{U}$ acted by a $\mathcal{U}$-gp $\mathcal{G}$ …

- $\mathcal{Y}$ becomes a (right) $\mathcal{G}$-torsor over $\mathcal{U}$ if
  (i) $\mathcal{Y} \to \mathcal{U}$ is an epi.
  (ii) $\mathcal{Y} \times_{\mathcal{U}} \mathcal{G} \to \mathcal{Y} \times_{\mathcal{U}} \mathcal{Y}$ is an iso.
  $(y, g) \longmapsto (y, yg)$

  If $\mathcal{Y} \to \mathcal{U}$ has a section, call $\mathcal{Y}$ trivial (i.e. $\mathcal{Y} \cong \mathcal{G}_r$ as $\mathcal{G}$-torsor over $\mathcal{U}$).
- Let $\mathcal{U} = e \in Sh(C)$ be a final obj $\rightsquigarrow$ $\mathcal{G}$-torsor over C, denoted by $\mathrm{Tors}(C, \mathcal{G})$.
  $H^1(C, \mathcal{G}) = \mathrm{Tors}(C, \mathcal{G})/\cong$ a pt set.

2.2 Fibred cats with topologies.

- A fibred cat (in groupoid) over $\mathcal{C}$ is a functor $p : \mathcal{T} \to \mathcal{C}$ s.t. for any $U' \xrightarrow{f} U$ in $\mathcal{C}$ and $T \in \mathcal{T}$, $p(T) = U$, there is a "pull-back" $f^*T \to T$, picture

$$\begin{tikzcd} f^*T \arrow[r] \arrow[d, no head, squiggly] & T \arrow[d, no head, squiggly, "p"] \\ U' \arrow[r, "f"] & U \end{tikzcd}$$

and any fiber cat $\mathcal{T}_U$ is a groupoid.
  $\mathcal{T}$ inherited canonically the topology of $\mathcal{C}$.
- A stack (in groupoid) over $\mathcal{C}$ is $\mathcal{Y} \to \mathcal{C}$ s.t. "descent is effective", and we have

(2.3)
$$\begin{tikzcd} & Sh(C) \arrow[r, "y_C", "\simeq"'] \arrow[d, hook] & \mathrm{Stk}_{set}/C \arrow[r, hook] \arrow[d, hook] & \mathrm{Stk}_{sh}/C \arrow[r, hook, dashed] & \mathrm{Stk}/C \\ C \arrow[r, "\varepsilon_C"] & Psh(C) \arrow[r, "y_C", "\simeq"'] & \mathrm{Fib}_{set}/C \arrow[r, hook] & \mathrm{Fib}_{psh}/C \arrow[r, hook, dashed] \arrow[u, dashed, "\text{ess im}"] & \mathrm{Fib}/C \arrow[u, dashed, "\text{strict full}"'] \end{tikzcd}$$

fibred in Set / setoids

- A stack $\mathcal{Y} \to \mathcal{C}$ is a gerbe over C rep by psh / sh if
  (i) $\forall U \in \mathcal{C}$, $\exists$ cov. $\{U_i \to U\}$ s.t. $\mathcal{Y}_{U_i} \neq \emptyset$,
  (ii) — — $x, y \in \mathcal{Y}_U$, $\exists$ cov. $\{U_i \to U\}$ s.t. $x|_{U_i} \cong y|_{U_i}$ in $\mathcal{Y}_{U_i}$.

  If $\mathcal{Y} \to \mathcal{C}$ has a sec, call it trivial.
- $\mathcal{Y} \to \mathcal{C}$ a gerbe bound by $\mathcal{G} \in Ab(\mathcal{C})$. $H^2(\mathcal{C}, \mathcal{G}) =$
  $\mathrm{Gerb}(\mathcal{C}, \mathcal{G})/\mathcal{G}\text{-equiv}$ a pt set.

2.3 Topologies

Now let $\mathcal{C} = S_\tau = (Sch/S)_\tau$, $\tau \in \{Et, fppf\}$ (big), and we mainly working in Fib/S, Stk/S_τ, etc. Look at (2.3), we have ε_{S_τ} factors through $Sh(S_\tau)$ …

Thus schemes, torsors, stacks, gerbes over S_τ (OLP3)
are all in Fib/S. Moreover,
for $T_\tau \in \mathrm{Fib}/S$, $Y \in \mathrm{Tors}(T_\tau, \mathcal{G})$, $Y_\tau(Y) \in \mathrm{Fib}/T$,
denoted by $f: Y \to T$ $\Rightarrow (Y \to T \to S) \in \mathrm{Fib}/S$.
Fix $p: X \to S$ and $q: A \to S \in \mathrm{Fib}/S$. (Take $S = \mathrm{Spec}\, k$, $X = X$, $A \to S$ be $\mathrm{Spec}\, A_k \to \mathrm{Spec}\, k$. $\Rightarrow X(A) = X(A_k)$, $X(S) = X(k)$.)

2.4. Cohomology. Let $T \in \mathrm{Fib}/S$, and write

$$H^i_\tau(T, -) = H^i(T_\tau, -) : \mathrm{Ab}(T_\tau) \to \mathrm{Ab}$$

Facts.
- Fix $\mathcal{G} \in \mathrm{Ab}(S_\tau)$, $H^i_\tau(-, \mathcal{G}) : (\mathrm{Fib}/S)^\circ \to \mathrm{Set}$ is "stable". $\rightsquigarrow$ take $F = H^i_\tau(-, \mathcal{G})$ as 1.7, $\rightsquigarrow X(A)^{H^i_\tau(X, \mathcal{G})}$ and
$$X(A)^{i\text{-desc}} := \bigcap_{\mathcal{G} \in \mathrm{Ab}(S_\tau)} X(A)^{H^i_\tau(X, \mathcal{G})}$$
- $H^i_\tau(T, \mathcal{G}) \xrightarrow{\sim} \check{H}^i_\tau(T, \mathcal{G})$ if T has a final ob.
$H^1_\tau(T, \mathcal{G}) \xrightarrow{\sim} H^1_\tau(T, \mathcal{G})$ if $\mathcal{G} \in \mathrm{Sgrp}(T_\tau)$
$H^2_\tau(T, \mathcal{G}) \xrightarrow{\sim} H^2_\tau(T, \mathcal{G})$ if $\mathcal{G} \in \mathrm{Ab}(T_\tau)$ if — — —

2.5. Prop (Generalized descent by torsors). Define descent cat to be $X(A)^{\mathrm{desc}} = \bigcap_{\mathcal{G} \in \mathrm{Sgrp}(S_{fppf})} X(A)^{H^1_{fppf}(X, \mathcal{G})}$

- In the classical case, $X(A_k)^{\mathrm{desc}} \subseteq X(A_k)^{\mathrm{desc}}$
- If $A \in \mathrm{Fib}_{set}/S$, $X \in \mathrm{Fib}_{psh}/S$, $\mathcal{G} \in \mathrm{SGrp}(S_\tau)$, $f: Y \to X \in \mathrm{Tors}(X_\tau, \mathcal{G})$, then
$$X(A)^f = \bigcup_{\sigma \in \check{H}^1_\tau(S, \mathcal{G})} f^\sigma(Y^\sigma(A)).$$
$$\Rightarrow X(A)^{\mathrm{desc}} = \bigcap_{\mathcal{G},\ f: Y \to X \in \mathrm{Tors}(X_{fppf}, \mathcal{G})} \bigcup_\sigma f^\sigma(Y^\sigma(A))$$
and $X(S) = \bigcup_\sigma f^\sigma(Y^\sigma(S))$.

2.6 Thm (Descent by gerbes) Define 2-desc = 2-desc$_{et}$.
If $A \in \mathrm{Stk}_{set}/S_\tau$, $X \in \mathrm{Stk}/S_\tau$, $\mathcal{G} \in \mathrm{Ab}(S_\tau)$, $f: Y \to X \in \mathrm{Gerb}(X_\tau, \mathcal{G})$, then
$$X(A)^f = \bigcup_{\sigma \in H^2_\tau(S, \mathcal{G})} f^\sigma(Y^\sigma(A))$$

OL p4

$$\Rightarrow \mathcal{X}(A)^{2\text{-desc}} = \bigcap_{\mathcal{G},\ f:\mathcal{Y}\to\mathcal{X}\in \mathrm{Gerb}(\mathcal{X}_{et},\mathcal{G})} \bigcup_\sigma f^\sigma(\mathcal{Y}^\sigma(A)).$$

$$\mathcal{X}(S) = \bigcup_\sigma f^\sigma(\mathcal{Y}^\sigma(S))$$

2.7 $\mathcal{X}(A)^{2\text{-ndesc}} = \bigcap_{\mathcal{G}\in \mathrm{Sgrp}(S_{et})} \mathcal{X}(A)^{H^2_{et}(\mathcal{X},\mathcal{G})}$ "non ab 2-desc"

§3. Composite obstructions.

3.1 Def • Let $\mathbb{I} \hookrightarrow \mathrm{Fib}/S$ full sub 2-cat, ob map sending each $\mathcal{X}\in\mathbb{I}$ to an "ob cat", call ob map on $\mathbb{I}$, ~~it~~ is functorial on $\mathbb{I}$ if $\forall$ 1-mor $f:\mathcal{Y}\to\mathcal{X}$ in $\mathbb{I}$, we have $f(\mathcal{Y}(A)^{ob}) \subseteq \mathcal{X}(A)^{ob}$

• For functorial map ob on Fib/S (resp. Stk/S_{et}) define

$$\mathcal{X}(A)^{\mathrm{desc},ob} = \bigcap_{\substack{\mathcal{G}\in \mathrm{Sgrp}(S_{fppf})\\ f:\mathcal{Y}\to\mathcal{X}\in \mathrm{Tors}(\mathcal{X}_{fppf},\mathcal{G})}} \bigcup_{\sigma\in H^1_{fppf}(S,\mathcal{G})} f^\sigma(\mathcal{Y}^\sigma(A)^{ob})$$

$$\text{(resp. } \mathcal{X}(A)^{2\text{-desc},ob} = \bigcap_{\substack{\mathcal{G}\in Ab\\ f:\mathcal{Y}\to\mathcal{X}\in \mathrm{Gerb}}} \bigcup_{\sigma\in H^2_{et}} f^\sigma(\mathcal{Y}^\sigma(A)^{ob})).$$

3.2 Thm. Let $\delta\in\{\mathrm{desc}, 2\text{-desc}\}$, $A\in \mathrm{Fib}_{Set}/S$ (resp. $\mathrm{Stk}_{Set}/S_{et}$) and $\mathbb{I} = \mathrm{Fib}_{fsh}/S$ (resp. Stk/S_{et}) if $\delta = \mathrm{desc}$ (resp. 2-desc). Let $\mathcal{X}\in\mathbb{I}$ and ob functorial on $\mathbb{I}$.

• $\mathcal{X}(A)^{\delta,ob}$ is an ob cat and (δ, ob) is ~~f~~ also functorial on $\mathbb{I}$.

• $$\mathcal{X}(A)^{\delta,ob} \subseteq \mathcal{X}(A)^\delta \cap \mathcal{X}(A)^{ob}.$$

3.3 Cor. With ~~notation~~ ~~and~~ ass in 3.2, we have

$$\mathcal{X}(S) \subseteq \cdots \subseteq \mathcal{X}(A)^{\delta^{n+1},ob} \subseteq \mathcal{X}(A)^{\delta^n,ob} \subseteq \cdots \subseteq \mathcal{X}(A)^{\delta^1,ob}$$

$$\subseteq \mathcal{X}(A)^\delta \cap \mathcal{X}(A)^{ob} \subseteq \mathcal{X}(A),\ \text{all fuctoral in } \mathbb{I}.$$

3.4 Rk. We already obtain $(\delta^n, \mathrm{desc})$, (desc', ob) are not longer than $\mathcal{X}(A_k)^{\mathrm{desc}}$.

§4. The derived obstruction

4.1. Obs under a product.

- (Harpaz Schlank 13) $(X \times_k Y)(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} = X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} \times Y(\mathbb{A}_k)^{\text{ét},\mathrm{Br}}$
- (Sk, Zarhin, 14, Lc 20) $(X \times_k Y)(\mathbb{A}_k)^{\mathrm{Br}} = \text{------}$

Q. Can we obtain smaller obs also stable under a prod?

4.2. Modified 2-cat of $Sh(S_\tau)$-topoi. For a 1-mor $f : T' \to T$ in $\mathbb{P}\mathrm{ib}/S$, we have the corresp 1-mor of $Sh(S_\tau)$-topoi

$$\begin{tikzcd} Sh(T'_\tau) \arrow[rr, "f"] \arrow[dr, "x"'] & \arrow[d, phantom, "\Leftarrow\alpha"] & Sh(T_\tau) \arrow[dl, "y"] \\ & Sh(S_\tau) & \end{tikzcd}$$

i.e. 2-iso (of topoi) $\alpha : yf \Rightarrow x$, where $f = (f^*, f_*)$, ...

Not $\alpha^* = \mathrm{id}_{x^*} : x^* = f^* y^*$ $\rightsquigarrow$ modified $Sh(S_\tau)$-topoi, denoted by $\mathrm{Tps}/Sh(S_\tau)$.

4.3 Def (Derived obs) Define $\mathcal{X}(A)^{Sh_\tau}$ by

$x \in \mathcal{X}(A)^{Sh_\tau} \iff$ In $\mathrm{Tps}/Sh(S_\tau)$, $\exists$ 1-mor $x_0 : Sh(S_\tau) \to Sh(\mathcal{X}_\tau)$ along with a 2-iso $x \Rightarrow x_0 q$.

$\Rightarrow x^* \Rightarrow q^* x_0^* :$
$Sh(\mathcal{X}_\tau) \to Sh(A_\tau)$
SGrp
$\mathrm{Mod}(\mathcal{X}_\tau, \Lambda)$
$D(\mathcal{X}_\tau, \Lambda)$

and (comparative) $x_* \Rightarrow q_* x_{0*}$ ——

4.4. Prop. We have $\mathcal{X}(A)^{Sh_{fppf}} \subseteq \mathcal{X}(A)^{desc}$

$\mathcal{X}(A)^{Sh_\tau} \subseteq \mathcal{X}(A)^{i\text{-desc}_\tau}$, $\forall i \geq 0$

4.5 Thm (Product) If in the 2-cartesian diag in $\mathbb{P}\mathrm{ib}/S$.

$$\begin{tikzcd} U \arrow[r, "i'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ {} & {} \end{tikzcd}$$

$\mathcal{U}$ and f are "representable": then we have a 2-comm. diag in Cat.

$$\begin{tikzcd} V(A)^{sh_\tau} \arrow[r, "i'"] \arrow[d, "f'"] & Y(A)^{sh_\tau} \arrow[d, "f"] \\ U(A)^{sh_\tau} \arrow[r, "i"] & X(A)^{sh_\tau} \end{tikzcd}$$

that is, projections to U and X

induce a natural equiv in Cat

$$V(A)^{sh_\tau} \xrightarrow{\simeq} U(A)^{sh_\tau} \times_{X(A)^{sh_\tau}} Y(A)^{sh_\tau}$$

4.6 Cor Let X, Y k-schemes, then
$$(X \times_k Y)(\mathbb{A}_k)^{sh_\tau} = X(\mathbb{A}_k)^{sh_\tau} \times Y(\mathbb{A}_k)^{sh_\tau}.$$

4.7 Rk. taking $\tau = fppf$ in 4.6, we obtain (4.4, 2.5) same prod formula as mentioned in 4.1 but with dev ob $X(\mathbb{A}_k)^{sh_{fppf}}$ not larger than all known obs before this work.

□