Rational points on varieties: an introduction

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§1. Introduction

1.1. We are interested in whether equations have rational solutions. Suppose $f_i(X_1, \cdots X_n) \in \mathbb{Q}[X_1, \cdots X_n]$.

$$(1.2)\quad \begin{cases} f_1(X_1, \cdots, X_n) = 0 \\ \vdots \\ f_m(X_1, \cdots, X_n) = 0 \end{cases}$$ has solution ~~on~~ in $\mathbb{Q}$?

1.3. Global and local solutions. We know that $\mathbb{Q}$ is contained in all its completions $\mathbb{Q}_p$, $p = 2, 3, \cdots$ prime, and $\mathbb{R}$. Then (1.2) is solvable in $\mathbb{Q}$ (global solution)

$\Rightarrow$ it is solvable in all $\mathbb{Q}_p$ and $\mathbb{R}$ (local solution).

Thus we first look at local solutions of it, ~~of which~~ is simpler, e.g. Hensel's Lemma) and then investigate its global ones.

Then question is, how?

§2

§2. Rational points on varieties.

2.1. The language of arithmetic geometry.

Let $X = \mathrm{Spec}\left(\mathbb{Q}[X_1, \cdots X_n] / (f_1, \cdots f_m)\right)$. By abusing notation, we will write R for $\mathrm{Spec}\, R$. The solution of (1.2) on $\mathbb{Q}$ is in one-to-one correspondence with $X(\mathbb{Q}) := \mathrm{Hom}_{\mathrm{Spec}\,\mathbb{Q}}(\mathrm{Spec}\,\mathbb{Q}, X)$

$= \mathrm{Hom}_{\mathbb{Q}\text{-alg}}\left(\mathbb{Q}[X_1, \cdots X_n] / \langle f_1, \cdots f_m \rangle, \mathbb{Q}\right)$.

Since $\mathbb{Q} \subset \prod_p \mathbb{Q}_p \times \mathbb{R}$, in fact, $\mathbb{Q} \subset \underbrace{\mathbb{A}_{\mathbb{Q}}}_{\text{adèle ring}}$

Then we have $X(\mathbb{Q}) \subseteq X(\mathbb{A}_{\mathbb{Q}})$, and then $X(\mathbb{Q}) \neq \emptyset \Rightarrow X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$

This is a reformulation of 1.3.

2.2. Rational and adèlic points on varieties

We now make a general setting.

2.3. Def (i) Let k be a field. A variety over k is a seperated k-scheme of finite type.

(ii). If k is a global field (number field for this talk), we call $X(k)$ (resp $X(\mathbb{A}_k)$) the set of rational (resp. adèlic poits) of X.

2.4. Remark. (i) If X is proper (say, projective). then $X(\mathbb{A}_k) = \prod_{v \in \Omega_k} X(k_v)$.

(ii). $X(\mathbb{A}_k)$ has a natural topology, and one also consider approximations (not this talk).

2.5. Clearly we have $X(k) \subseteq X(\mathbb{A}_k)$. Thus $X(k) \neq \emptyset \Rightarrow X(\mathbb{A}_k) \neq \emptyset$. What about the converse?

2.6. Def. If $X(\mathbb{A}_k) \neq \emptyset \Rightarrow X(k) \neq \emptyset$, we say Hasse principal holds for X/k

2.7 Thm (Hasse-Min) For X/k defined by quadratic forms, Hasse principal always holds.

2.8. Example. For $X: 3X^3 + 4Y^3 + 5Z^3 = 0 \ /\mathbb{Q}$, Hasse principal not holds. In fact, this is the case for many varieties, so we want to look for sets between $X(k)$ and $X(\mathbb{A}_k)$.

2.9. The F-obstruction [Poonen 7]. Let $F: (\mathrm{Sch}/k)^{\circ} \to \mathrm{Set}$ be a functor and T a k-scheme. For any T-point $x \in X(T)$, and $A \in F(X)$, the evaluation of A at x is the image of x under the pull-back map $F(X) \xrightarrow{x^*} F(T)$ induced by $T \xrightarrow{x} X$, denoted by $A(x)$.

$$\begin{array}{ccccc} X & & F(X) & & A \\ \uparrow x & & \downarrow x^* & & \downarrow \\ T & & F(T) & & A(x) \end{array}$$

Now we have an obvious comm. diagram.

(2.10).
$$\begin{array}{ccc} X(k) & \hookrightarrow & X(\mathbb{A}_k) \\ \downarrow A(-) & & \downarrow A(-) \\ F(k) & \longrightarrow & F(\mathbb{A}_k) \end{array}$$

From this, if we define

$$X(\mathbb{A}_k)^A := \{ P \in X(\mathbb{A}_k) \mid A(P) \in \operatorname{im}(F(k) \to F(\mathbb{A}_k)) \}$$

then $X(k) \subseteq X(\mathbb{A}_k)^A$, i.e., we obtain constraints on the locus in $X(\mathbb{A}_k)$ where ~~k~~-rational points can lie.

2.11 ~~Def~~. The set $X(\mathbb{A}_k)^A$ is called the obstruction defined by A. Imposing all constraints made by $A \in F(X)$ yields the subset

$$X(\mathbb{A}_k)^F = X(\mathbb{A}_k)^{F(X)} := \bigcap_{A \in F(X)} X(\mathbb{A}_k)^A,$$

called the F-set. Now we have

$$\emptyset \subseteq X(k) \subseteq X(\mathbb{A}_k)^F \subseteq X(\mathbb{A}_k)^A \subseteq X(\mathbb{A}_k)$$

2.12 Def. (i) We say there is an F-obstruction to the local-global principal if $X(\mathbb{A}_k) \neq \emptyset$ but $X(\mathbb{A}_k)^F = \emptyset$. In this case, clearly $X(k) = \emptyset$.

(ii). We say the F-obstruction is the only one to local-global principal if

$$X(\mathbb{A}_k)^F \neq \emptyset \Rightarrow X(k) \neq \emptyset.$$

2.13. Example. Let $F = \mathrm{Br} = H^2_{et}(-, \mathbb{G}_m)$ the Brauer-Grothendieck group, we obtain the Brauer-Manin set (obstruction) $X(\mathbb{A}_k)^{\mathrm{Br}}$. This is equivalent to [Ces 15], the usual definition:

(2.14) $$\langle -,- \rangle_{BM} : X(\mathbb{A}_k) \times \mathrm{Br}\, X \longrightarrow \mathbb{Q}/\mathbb{Z} \quad (\text{the Brauer-Manin pairing})$$

$$(P_v)_v,\ A \longmapsto \sum_v \operatorname{inv}_v A(P_v)$$

$X(\mathbb{A}_k)^{\mathrm{Br}} = \{ (P_v)_v \in X(\mathbb{A}_k) \mid \langle (P_v)_v, A \rangle_{\mathrm{BM}} = 0$ for all $A \in \mathrm{Br}\, X \}$.

For some classes of varieties, B-M ob is the only one, [Sum., CT, Sk, ...] and
[Sk99] $\exists$ $\mathbb{Q}$-variety $X/\mathbb{Q}$ s.t. $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} \neq \emptyset$ but $X(\mathbb{Q}) = \emptyset$

2.15 Example. Let G be a affine k-group (may non-comm), and

$F = \check{H}^1_{fppf}(-, G)$ the first Čech cohomology (a pointed set) in fppf topology. If G is comm, then

$$\check{H}^1_{fppf}(X, G) = H^1_{fppf}(X, G) \underset{\substack{\uparrow \\ \mathrm{char}\, k = 0,\ G \text{ smooth}}}{=} H^1_{\acute{e}t}(X, G).$$

We have the subset $X(\mathbb{A}_k)^{\check{H}^1_{fppf}(-, G)}$.

To obtain smaller subset, we use all G.

2.16 Def. The descent set (obstruction) is

$$X(\mathbb{A}_k)^{desc} = \bigcap_{\text{all affine } k\text{-gp } G} X(\mathbb{A}_k)^{\check{H}^1_{fppf}(-, G)}.$$

2.17 Prop. [Sk01] For X regular, quasi-projective,

$$X(\mathbb{A}_k)^{desc} \subseteq X(\mathbb{A}_k)^{\mathrm{Br}} = \bigcap_{n \geq 1} X(\mathbb{A}_k)^{\check{H}^1_{fppf}(-, PGL_n)}$$

Can we find smaller subset?

2.18. Def. An X-torsor under an X-group scheme G is a X-scheme Y with an action of G compatible with the projection to X, and s.t. $Y \to X$ is fppf, and $Y \times_X G \to Y \times_X Y$ is iso.
$(y, s) \longmapsto (y, ys)$

(By Giro's fppf descent theory, this is to say
$\exists$ fppf covering $(U_i \to X)_i$ s.t. $Y_{U_i} \curvearrowleft G_{U_i}$
$\cong$ $G_{U_i} \curvearrowleft G_{U_i}$). $G \curvearrowleft G$ is called trivial torsor.
Then X-torsor under $G/_{\mathrm{iso}} \overset{1:1}{\longleftrightarrow} \check{H}^1(X, G)$ as pointed set.

The descent by torsors says.

2.19. Prop. Let G aff k-gp and $f: Y \xrightarrow{G} X$ a G-torsor. Then

(2.20) $$X(\mathbb{A}_k)^{[Y]} \; (= [Y] \in \check{H}^1_{fppf}(X, G)) = X(\mathbb{A}_k)^f = \bigcup_{\sigma \in H^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k)).$$

where $f^\sigma : Y^\sigma \xrightarrow{G^\sigma} X$ is the torsor twisted by $\sigma \in H^1(k, G) \to \check{H}^1_{fppf}(X, G)$.

2.21. Thus by (2.20) we have

$$X(\mathbb{A}_k)^{desc} = \bigcap_{\substack{f: Y \xrightarrow{G} X \text{ all torsors} \\ \text{all aff } k\text{-gp } G}} \bigcup_{\sigma \in H^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k))$$

and $$X(k) = \dot{\bigcup_{\sigma \in H^1(k,G)}} f^\sigma(Y^\sigma(k)).$$

This lead to the smaller subset.

2.22 Ref (i) [Po10, Rem 09, Sk 09] The étale-Brauer set is

$$X(\mathbb{A}_k)^{et, Br} := \bigcap_{\substack{\text{finite } k\text{-gp } G, \\ \text{all } f: Y \xrightarrow{G} X}} \bigcup_{\sigma \in H^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k)^{Br})$$

(ii). The iterated descent set is

$$X(\mathbb{A}_k)^{desc, desc} := \bigcap_{\substack{\text{all affine } G, \\ \text{all } f: Y \xrightarrow{G} X}} \bigcup_{\sigma \in H^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k)^{desc})$$

2.23 Thm. [Sk 09, St 007, Rem 09, CDX 16]. Let X smooth, q-p, geo. integral variety. Then

$$X(\mathbb{A}_k)^{desc} = X(\mathbb{A}_k)^{et, Br}.$$

2.24 Thm [Cao 20]. X as 2.23. Then

$$X(\mathbb{A}_k)^{desc} = X(\mathbb{A}_k)^{desc, desc} = X(\mathbb{A}_k)^{desc, desc, desc} \cdots$$

up till now, no subset smaller than descent set is found.

§3. Brauer-Manin set under a product

In this section, we restrict ourself ~~to~~ smooth geo. int. varieties over a ~~number~~ field k. $g_k := \mathrm{Gal}(\bar{k}/k)$

~~All~~ Cohomologies are all étale.

3.1. The B-M set is good for calculation, and a natural question is

$$(X \times_k Y)(\mathbb{A}_k)^{\mathrm{Br}} \overset{?}{=} X(\mathbb{A}_k)^{\mathrm{Br}} \times Y(\mathbb{A}_k)^{\mathrm{Br}}.$$

write $X \times Y$ for simplicity

3.2 Thm [SZ 14] For X proper, $(X\times Y)(\mathbb{A}_k)^{\mathrm{Br}} = X(\mathbb{A}_k)^{\mathrm{Br}} \times Y(\mathbb{A}_k)^{\mathrm{Br}}$

3.3. The idea. $\subseteq$ is clear by functoriality.

For the converse, [Gro 68] says Br is torsion. Thus the following two lemmas ~~are~~ enough

3.4 Lemma.

$$H^2(X, \mu_n) \oplus H^2(Y, \mu_n) \oplus \mathrm{Hom}_k(S_X, S_Y) \xrightarrow{(p_X^*, p_Y^*, \varepsilon)} H^2(X\times Y, \mu_n), \quad \text{where}$$

$S_X^* := \mathrm{Hom}_{k\text{-grps}}(S_X, \mathbb{G}_m) = H^1(\bar{X}, \mu_n)$

S_Y^*

and $\varepsilon : \mathrm{Hom}_k(S_Y, S_Y^*) \to H^2(X\times Y, \mu_n)$

$\phi \longmapsto \phi_*[T_X] \cup [T_Y]$.

[Cao 20] universal torsors of n-torsion type

3.5 Lemma

$\forall\ n \geq 1$

$$X(\mathbb{A}_k)^{(\mathrm{Br}_1 X)_n} \times Y(\mathbb{A}_k)^{(\mathrm{Br}_1 Y)_n} \subseteq (X\times Y)(\mathbb{A}_k)$$

where define $0 \to \mathrm{Br}_1 X \to \mathrm{Br}\, X \to (\mathrm{Br}\, \bar{X})^{g_k}$

$\to \mathrm{Br}_2 X \to 0$

3.6. For open varieties? (non-proper)

3.4 is still correct. But 3.5 needs to be modified.

3.7 Lemma. Assume $X(\mathbb{A}_k)^{Br}$, and $Y(\mathbb{A}_k)^{Br}$ (this condition is not essential) are both nonempty. Then

$$H^2(X,\mu_n)\oplus H^2(Y,\mu_n)\oplus \mathrm{Hom}_k(S_X, S_Y^*) \xrightarrow{(p_X^*,\, p_Y^*,\, \varepsilon)} H^2(X\times Y,\mu_n)$$

is surjective.

3.8. The key idea is to extend the low term exact seq associated to the spectral seq $E_2^{pq}(X) = H^p(k, H^q(X,\mu_n)) \Rightarrow E^{p+q}(X) = H^{p+q}(X,\mu_n)$,

$$E_2^{2,0}(X) \to \underset{E_1^2(X)}{\mathbb{H}^2(k, \tau_{\le 1}\mathbb{R}p_*\mu_n)} \to E_2^{1,1}(X) \to E_2^{3,0}(X) \to \mathbb{H}^3(k, \tau_{\le 1}\mathbb{R}p_*\mu_n),$$

then truncate it to

$$E_2^{2,0}(X) \to E_1^2(X) \to E_2^{1,1}(X) \to 0.$$

3.9 Thm [Lv 20] 3.2 Thm is correct for any variety.

3.10 Remark. [HS 13] $(X\times Y)(\mathbb{A}_k)^{\acute{e}t,Br} = X(\mathbb{A}_k)^{\acute{e}t,Br}\times Y(\mathbb{A}_k)^{\acute{e}t,Br}$ for any varieties.

§A. Brauer gp under a product

One may also interstiget the natral map $\mathrm{Br}\, X \oplus \mathrm{Br}\, Y \xrightarrow{(p_X^*, p_Y^*)} \mathrm{Br}\,(X \times Y)$

Then under some extra conditions, [Ga18] says

A.1. Thm. (Some conditions). $0 \to \mathrm{Br}_? k \to \mathrm{Br}\, X \oplus \mathrm{Br}\, Y$
$\to \mathrm{Br}\,(X \times Y) \to \mathrm{Br}\,(\bar{X} \times \bar{Y})^{g_k} / (\mathrm{Br}\, \bar{X})^{g_k} \oplus (\mathrm{Br}\, \bar{Y})^{g_k}$
$\to \mathrm{Br}_2\,(X \times Y) / \mathrm{Br}_2 X \oplus \mathrm{Br}_2 Y \to 0$.

A.2 Thm [Lu20] For smth, geo. int., q-p varieties over a field k,
$\mathrm{cok}(\,(\mathrm{Br}\,\bar{X})^{g_k} \oplus (\mathrm{Br}\,\bar{Y})^{g_k} \to \mathrm{Br}\,(\bar{X} \times \bar{Y})^{g_k}\,)$,
$\mathrm{cok}\,(\,\mathrm{Br}_1 X \oplus \mathrm{Br}_1 Y \to \mathrm{Br}_1\,(X \times Y)\,)$
$\mathrm{cok}\,(\,\mathrm{Br}\, X \oplus \mathrm{Br}\, Y \to \mathrm{Br}\,(X \times Y)\,)$ are finite.

A.3. Remark. If restected in projective varieties, this was first showed by [SZ14].

§B. Abelian descent for open varieties

(to be continue ...)

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