The Brauer-Manin Obstruction on Algebraic Stacks

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Joint work with Han Wu

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Outline

On integer factorization

- Let $N = pq$ be an RSA modulus. Want to factor it . . .
- Let $E_{r,N} : y^2 = x^3 - 2rNx$ elliptic curve over $\mathbb{Q}$. It is known that:
  - can find a very small r such that $E_{r,N}$ has rank one (easy), and
  - can use a non-torsion point (hard) $P \in E_{r,N}(\mathbb{Q})$ to factor N (just gcd).
- The problem reduces to find P: algorithm to find rational points on varieties.

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- Basically, for k number fields, it contained in adèlic points $X(\mathbf{A}_k)$.
- Conversely, $X(\mathbf{A}_k) \neq \emptyset$ implies $X(k) \neq \emptyset$?

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- People constructed some subsets $X(k) \subseteq X(\mathbf{A}_k)^{\text{obs}} \subseteq X(\mathbf{A}_k)$ called obstructions.
  - For example, obs can be Br (Brauer-Manin), (ét, Br) (étale-Brauer), desc (descent) ...
- Point is: for many X, $X(\mathbf{A}_k)^{\text{obs}} \neq \emptyset$ does implies $X(k) \neq \emptyset$!
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Why algebraic stacks

- Varieties are *Sch* (schemes). There are lager categories of geometric objects $Sch \subset Esp$ (algebraic spaces) $\subset \mathbb{C}$hp (algebraic stacks).
- To study descent theory on H^2-level, or moduli spaces classifying geo. obs.
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- "Functors" $\mathcal{X} : Sch^{\text{op}} \to Gpd$ (groupoids) that are both "sheaves" and "algebraic" ...
  - Simplest example: BG (classifying stack), $[X/G]$ (quotient stack) ...
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Example: a results on stacky curves

- Let k be number field, p, q primes, and
- $\mathcal{Y}_{(p,q)} = \mathrm{Proj}(\mathcal{O}_k[X, Y, Z]/(Z^2 - pX^2 - qY^2))$,
- acted on by μ_2: $[X : Y : Z] \mapsto [X : Y : -Z]$.

Theorem 2.1 ([WL23])

There exists infinite many (p, q) such that the stacky curve $\mathcal{X}_{(p,q)} = [\mathcal{Y}_{(p,q)}/\mu_2]$ violating local-global principle for integral pts.

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Brauer groups

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  - Simplest example: $\operatorname{Br}(\mathbb{C}) = \operatorname{Br}(\mathbb{F}_q) = 0$, $\operatorname{Br}(\mathbb{R}) = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$, $\operatorname{Br}(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z} \cdots$
- The functor Br creates $\mathcal{X}(\mathcal{A})^{\operatorname{Br}}$,

$$\mathcal{X}(\mathcal{A})^{\operatorname{Br}} = \{x \in \mathcal{X}(\mathcal{A}) \mid x^* \operatorname{Br}\mathcal{X} \subseteq \operatorname{im}(\operatorname{Br} k \to \operatorname{Br}\mathbf{A}_k)\}.$$

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Obstructions made by functors

- In general, let $q : \mathbf{A}_k \to \operatorname{Spec} k$, for any stable functor $F : (\mathbb{C}\mathrm{hp}/k)^{\mathrm{op}} \to \mathit{Set}$ and $A \in F(\mathcal{X})$,

$$\mathcal{X}(\mathbf{A}_k)^A = \{x \in \mathcal{X}(\mathbf{A}_k) \mid A(x) \in \operatorname{im} F(q)\},$$

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- Then

$$\mathcal{X}(k) \to \mathcal{X}(\mathbf{A}_k)^F \subseteq \mathcal{X}(\mathbf{A}_k)^A \subseteq \mathcal{X}(\mathbf{A}_k).$$

Remark

The map $\mathcal{X}(k) \to \mathcal{X}(\mathbf{A}_k)$ is not necessary injective. For example, let G be a linear k-group. Then

$$\ker(BG(k) \to BG(\mathbf{A}_k)) = \ker(H^1(k, G) \to \check{H}^1_{\mathrm{fppf}}(\mathbf{A}_k, G)) = Ш^1(G/k)$$

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Torsors

- Let G be a k-group, $\mathcal{X}/k \in \mathbb{C}\mathrm{hp}/k$, a G-torsor over $\mathcal{X}_{\mathrm{fppf}}$ is a sheaf $\mathscr{Y}$ on $\mathcal{X}_{\mathrm{fppf}}$ acted on by G, such that $G \times \mathscr{Y} \cong \mathscr{Y} \times \mathscr{Y}$.
  - Simplest example: $X^2 + Y^2 = n$ is a G-torsor over $\mathbb{Q}$ where $G : X^2 + Y^2 = 1$.
- They form a groupoid $\mathrm{Tors}(\mathcal{X}_{\mathrm{fppf}}, G)$.
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Torsors are algebraic

Lemma 2.2

- Any torsor $\mathscr{Y} \in \mathrm{Tors}(\mathcal{X}_{\mathrm{fppf}}, G)$ is algebraic, i.e., is in $\mathbb{C}\mathrm{hp}/k$.
- In particular, a 1-morphism of algebraic stacks $\mathcal{Y} \to \mathcal{X}$ is in $\mathrm{Tors}(\mathcal{X}_{\mathrm{fppf}}, G)$ if and only if $\mathcal{X} \xrightarrow{\sim} [\mathcal{Y}/G]$ is the quotient stack.

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Sansuc's exact sequence

- Let X be a k-var. and G a connected k-group acting on X.
- Let $\mathcal{Y} = [X/G]$ be the quo. stack, i.e., $f : X \to \mathcal{Y}$ is in $\mathrm{Tors}(\mathcal{Y}, G)$ (Lem. 2.2).
- Let $\mathrm{U} = \mathbf{G}_m/k^\times$, $\mathrm{Pic} = H^1_{\text{ét}}(-, \mathbf{G}_m) \in PSh(Sch/k)$.

Theorem 2.3 ([LW23])

Have the exact sequence

$$0 \to \mathrm{U}\,\mathcal{Y} \xrightarrow{f^*} \mathrm{U}\,X \to \mathrm{U}\,G \to \mathrm{Pic}\,\mathcal{Y} \xrightarrow{f^*} \mathrm{Pic}\,X \to \mathrm{Pic}\,G \to$$

$$\mathrm{Br}\,\mathcal{Y} \xrightarrow{f^*} \mathrm{Br}\,X \xrightarrow{\rho^* - p_2^*} \mathrm{Br}(G \times_k X),$$

where $\rho,\ p_2 : G \times_k X \to X$ is the action and projection.

- Extend classical one by Sansuc [San81].

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- Let $\mathrm{U} = \mathbf{G}_m/k^\times$, $\mathrm{Pic} = H^1_{\text{ét}}(-, \mathbf{G}_m) \in PSh(Sch/k)$.

Theorem 2.3 ([LW23])

Have the exact sequence

$$0 \to \mathrm{U}\,\mathcal{Y} \xrightarrow{f^*} \mathrm{U}\,X \to \mathrm{U}\,G \to \mathrm{Pic}\,\mathcal{Y} \xrightarrow{f^*} \mathrm{Pic}\,X \to \mathrm{Pic}\,G \to$$
$$\mathrm{Br}\,\mathcal{Y} \xrightarrow{f^*} \mathrm{Br}\,X \xrightarrow{\rho^* - p_2^*} \mathrm{Br}(G \times_k X),$$

where $\rho,\ p_2 : G \times_k X \to X$ is the action and projection.

- Extend classical one by Sansuc [San81].

Torsionness of the Brauer group

- Grothendieck [Gro68] showed that $\operatorname{Br} X$ is torsion for a regular scheme X.

Corollary 2.4 ([LW23])

Let $\mathcal{X} \in \mathbb{C}\mathrm{hp}/k$ which can be covered by finitely many open substacks $[X_i/G_i]$ where X_i is a smooth k-var. and G_i a linear k-group acting on X_i. Then $\operatorname{Br} \mathcal{X}$ is torsion.

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Antieau and Meier [AM20] showed that Brauer groups of regular Noetherian Deligne-Mumford stack are torsion.

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Outline

Fundamental exact sequence

- Let $D(k)$ be derived category of complexes of k-modules.
- Let S be a k-group of multiplicative type,
- whose Cartier dual $\hat{S}$ is a finitely generated k-module.
- Let $p : \mathcal{X} \to \operatorname{Spec} k$ be an alg. stack, and $\mathrm{KD}'(\mathcal{X})$ be cone of $\mathbf{G}_m[1] \to Rp_*\mathbf{G}_m[1]$ in $D(k)$.

Theorem 2.5

Have the fundamental exact sequence

$$0 \to H^1(k, S) \xrightarrow{p^*} H^1_{\mathrm{fppf}}(\mathcal{X}, S) \xrightarrow{\chi} \mathrm{Hom}_{D(k)}(\hat{S}, \mathrm{KD}'(\mathcal{X})) \xrightarrow{\partial} H^2(k, S) \xrightarrow{p^*} H^2_{\mathrm{fppf}}(\mathcal{X}, S),$$

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Descent

- Let $a \in H^1(k, \hat{S})$ and $\mathcal{X} \in \mathbb{C}\mathrm{hp}/k$.
- I [Lv22] gave a commutative diagram

$$\begin{array}{ccc} H^1(\mathcal{X}, S) & \xrightarrow{\chi} & \mathrm{Hom}_{D(k)}(\hat{S}, \mathrm{KD}'(\mathcal{X})) \\ \big\downarrow{\scriptstyle p^*(a)\cup -} & & \big\downarrow{\scriptstyle a\cup -} \\ \mathrm{Br}_1 \mathcal{X} & \xrightarrow{r} & H^1(k, \mathrm{KD}'(\mathcal{X})). \end{array}$$

- Let $f : \mathcal{Y} \to \mathcal{X}$ be an S-torsor. Define $\lambda = \chi([f])$, and
- $\mathrm{Br}_\lambda \mathcal{X} = r^{-1}(\lambda_*(H^1(k, \hat{S}))) \subseteq \mathrm{Br}_1 \mathcal{X}$.

Proposition 2.6 ([LW23])

We have $\mathcal{X}(\mathbf{A}_k)^f = \mathcal{X}(\mathbf{A}_k)^{\mathrm{Br}_\lambda}$

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The Brauer-Manin pairing

- Suppose that $\mathcal{X}$ is of finite type over k.

- As in classical case, the Brauer-Manin pairing for $\mathcal{X}$

$$\langle -, - \rangle_{\mathrm{BM}} : \mathcal{X}(\mathbf{A}_k) \times \operatorname{Br} \mathcal{X} \to \mathbb{Q}/\mathbb{Z},$$
$$((x_v)_v, A) \mapsto \sum_{v \in \Omega_k} \operatorname{inv}_v A(x_v),$$

  is well-defined,

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The Brauer-Manin pairing (a variant)

- Writing $\mathcal{X}_v = \mathcal{X} \times_k k_v$, one also defines

$$\text{Б}\mathcal{X} = \ker(\mathrm{Br}_a \mathcal{X} \to \prod_{v \in \Omega_k} \mathrm{Br}_a \mathcal{X}_v).$$

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Outline

Descent and second descent obstruction

Definition 2.7

The descent obstruction is

$$\mathcal{X}(\mathbf{A}_k)^{\mathrm{desc}} = \bigcap_{\text{linear } k\text{-group } G} \mathcal{X}(\mathbf{A}_k)^{\check{H}^1_{\mathrm{fppf}}(-,G)}.$$

We also define

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Comparison of obstructions

Theorem 2.8

Let X be a smooth alg. k-stack of f.t. that is either DM or Zariski-locally quo. of k-var. by a linear k-group. Then

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Descent for Brauer-Manin set along a torsor

- Let $\mathcal{X} = [Y/G]$ where Y is a smooth geometrically integral k-var. and G a conn. linear k-group acting on Y.
- Let $f : Y \to \mathcal{X}$ be the canonic map making Y a G-torosr over $\mathcal{X}$.
- One also defines invariant Brauer subgroup (as in Cao [Cao18]) to be

$$\mathrm{Br}_G \mathcal{X} = \{b \in \mathrm{Br}\,\mathcal{X} \mid \rho^* b - p_2^* b \in p_1^* \mathrm{Br}\, G\}$$

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Have

$$\mathcal{X}(\mathbf{A}_k)^{\mathrm{Br}} = \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y_\sigma(\mathbf{A}_k)^{\mathrm{Br}_{G_\sigma}(Y_\sigma)}).$$

- Key is to modify Sansuc exact seq. (Thm. 2.3) using Br_G.
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Known results

- Let k be a number field.
- The product preservation property of Brauer-Manin set was first established by Skorobogatov and Zarhin [SZ14] for smooth geo. int. projective k-vars.,
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Product of alg. stacks

Theorem 2.11

The functor

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Key ingredients of proof

- The torsionness of $\mathrm{Br}\,\mathcal{X}$ (Cor. 2.4).
- Existence of universal torsor of n-torsion
- Künneth formula for $H^i_{\text{ét}}(-,\mu_n)$ on $\overline{k}$, $i=1,2$:
  - Künneth for stacks $Rp_*K \boxtimes^L_\Lambda Rq_*L \cong R(p\times q)_*(K \boxtimes^L_\Lambda L)$ (coh. desc),
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Corollary 2.12

If $\mathcal{X}$ and $\mathcal{Y}$ are stacks quos. of smooth geo. int. k-varieties by conn. linear k-groups. Then

$$\mathcal{X}(\mathbf{A}_k)^{\mathrm{Br}} \times \mathcal{Y}(\mathbf{A}_k)^{\mathrm{Br}} \xrightarrow{\simeq} (\mathcal{X} \times_k \mathcal{Y})(\mathbf{A}_k)^{\mathrm{Br}}$$

Outline

Open problems

- Do we have

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  (it is true for varieties)

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Thanks for your attention.

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