Obstructions to local-global principle on algebraic stacks

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Outline

Rational points

- Let X/k variety, rational points is the set $X(k)$.
- Basically, for k number fields, it contained in adèlic points $X(\mathbf{A}_k)$.
- Conversely, $X(\mathbf{A}_k) \neq \emptyset$ implies $X(k) \neq \emptyset$?

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Obstructions

- People constructed some subsets $X(k) \subseteq X(\mathbf{A}_k)^{\text{obs}} \subseteq X(\mathbf{A}_k)$ called obstructions.
  - most obs comes from a functor F, namely,
  - for $F : (\mathcal{S}\mathrm{ch}_{/k})^{op} \to \mathcal{S}\mathrm{et}$, the map $X(k) \to X(\mathbf{A}_k)$ factorizes as $X(k) \to \mathrm{Map}(F(X), F(k)) \times_{\mathrm{Map}(F(X),F(\mathbf{A}_k))} X(\mathbf{A}_k) \xrightarrow{q^*} X(\mathbf{A}_k)$,
  - and we define $X(\mathbf{A}_k)^F = \mathrm{im}(q^*)$ $\quad (q : \mathrm{Spec}\,\mathbf{A}_k \to \mathrm{Spec}\,k)$.
- Point is: for many X, $X(\mathbf{A}_k)^{\text{obs}} \neq \emptyset$ does implies $X(k) \neq \emptyset$! And can help to find $P \in X(k)$.
- Example: Brauer-Manin obstruction $X(\mathbf{A}_k)^{\text{Br}}$, defined by Brauer group and computable.

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Algebraic stacks

- Varieties are $\mathcal{S}$ch (schemes). There are lager categories of geometric objects $\mathcal{S}\mathrm{ch} \subset \mathcal{E}\mathrm{sp}$ (algebraic spaces) $\subset \mathcal{C}\mathrm{hp}$ (algebraic stacks).
- Simplest example: BG (classifying stack), $[X/G]$ (quotient stack) . . .

Remark

The map $\mathcal{X}(k) \to \mathcal{X}(\mathbf{A}_k)$ is not necessary injective. For example, let G be a linear k-group. Then

$$\ker(BG(k) \to BG(\mathbf{A}_k)) = \ker(H^1(k, G) \to \check{H}^1_{\mathrm{fppf}}(\mathbf{A}_k, G)) = Ш^1(G/k)$$

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Relations between obstructons on varieties

- We have for smooth, quasi-projective geometrically integral k-variety X ([Har02, Poo17, Sto07, Sko09, Dem09, Poo10, HS13, CDX19, Cao20]),

$$X(\mathbf{A}_k)^{\mathrm{Br}} = X(\mathbf{A}_k)^{\mathrm{PGL}} = X(\mathbf{A}_k)^{\mathrm{conn}} = X(\mathbf{A}_k)^{2\text{-desc}} = X(\mathbf{A}_k)^{\mathbb{Z}h} \supseteq X(\mathbf{A}_k)^{h} = X(\mathbf{A}_k)^{\mathrm{desc,desc}} = X(\mathbf{A}_k)^{\mathrm{fin,desc}} = X(\mathbf{A}_k)^{\text{ét,Br}} = X(\mathbf{A}_k)^{\mathrm{desc}}.$$

Outline

Example: a results on stacky curves

- Let k be number field, p, q primes, and
- $\mathcal{Y}_{(p,q)} = \mathrm{Proj}(\mathcal{O}_k[X,Y,Z]/(Z^2 - pX^2 - qY^2))$,
- acted on by μ_2: $[X:Y:Z] \mapsto [X:Y:-Z]$.

Theorem 2.1 ([WL23])

There exists infinite many (p,q) such that the stacky curve $\mathcal{X}_{(p,q)} = [\mathcal{Y}_{(p,q)}/\mu_2]$ violating local-global principle for integral pts.

- The curves has genus $\frac{1}{2}$.
- Generalize Bhargava and Poonen [BP22].

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Brauer groups

- Brauer-Grothendieck group $\operatorname{Br}\mathcal{X} = H^2_{\text{ét}}(\mathcal{X}, \mathbf{G}_m)$ defined for every alg. stack $\mathcal{X}$. $\mathcal{X}(\mathcal{A})^{\operatorname{Br}}$.
  - Simplest example: $\operatorname{Br}(\mathbb{C}) = \operatorname{Br}(\mathbb{F}_q) = 0$, $\operatorname{Br}(\mathbb{R}) = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$, $\operatorname{Br}(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}\cdots$
- The functor Br creates $\mathcal{X}(\mathcal{A})^{\operatorname{Br}}$,

$$\mathcal{X}(\mathcal{A})^{\operatorname{Br}} = \{x \in \mathcal{X}(\mathcal{A}) \mid x^* \operatorname{Br}\mathcal{X} \subseteq \operatorname{im}(\operatorname{Br} k \to \operatorname{Br}\mathbf{A}_k)\}.$$

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Sansuc's exact sequence

- Let X be a k-var. and G a connected k-group acting on X.
- Let $\mathcal{Y} = [X/G]$ be the quo. stack, i.e., $f : X \to \mathcal{Y}$ is in $\mathrm{Tors}(\mathcal{Y}, G)$.
- Let $\mathrm{U} = \mathbf{G}_m/k^\times$, $\mathrm{Pic} = H^1_{\text{ét}}(-, \mathbf{G}_m) \in \mathcal{P}(\mathcal{S}\mathrm{ch}_{/k})$.

Theorem 3.1 ([LW23])

Have the exact sequence

$$0 \to \mathrm{U}\,\mathcal{Y} \xrightarrow{f^*} \mathrm{U}\,X \to \mathrm{U}\,G \to \mathrm{Pic}\,\mathcal{Y} \xrightarrow{f^*} \mathrm{Pic}\,X \to \mathrm{Pic}\,G \to$$
$$\mathrm{Br}\,\mathcal{Y} \xrightarrow{f^*} \mathrm{Br}\,X \xrightarrow{\rho^* - p_2^*} \mathrm{Br}(G \times_k X),$$

where $\rho,\ p_2 : G \times_k X \to X$ is the action and projection.

- Extend classical one by Sansuc [San81].

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Torsionness of the Brauer group

- Grothendieck [Gro68] showed that $\operatorname{Br} X$ is torsion for a regular scheme X.

Corollary 3.2 ([LW23])

Let $\mathcal{X} \in \mathcal{C}\mathrm{hp}_{/k}$ which can be covered by finitely many open substacks $[X_i/G_i]$ where X_i is a smooth k-var. and G_i a linear k-group acting on X_i. Then $\operatorname{Br} \mathcal{X}$ is torsion.

Remark

Antieau and Meier [AM20] showed that Brauer groups of regular Noetherian Deligne-Mumford stack are torsion.

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Fundamental exact sequence

- Let $D(k)$ be derived category of complexes of k-modules.
- Let S be a k-group of multiplicative type,
- whose Cartier dual $\hat{S}$ is a finitely generated k-module.
- Let $p : \mathcal{X} \to \operatorname{Spec} k$ be an alg. stack, and $\mathrm{KD}'(\mathcal{X})$ be cone of $\mathbf{G}_m[1] \to Rp_*\mathbf{G}_m[1]$ in $D(k)$.

Theorem 3.3 ([LW23])

Have the fundamental exact sequence

$$0 \to H^1(k, S) \xrightarrow{p^*} H^1_{\mathrm{fppf}}(\mathcal{X}, S) \xrightarrow{\chi} \mathrm{Hom}_{D(k)}(\hat{S}, \mathrm{KD}'(\mathcal{X})) \xrightarrow{\partial} H^2(k, S) \xrightarrow{p^*} H^2_{\mathrm{fppf}}(\mathcal{X}, S),$$

where the map χ is the extended type.

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Comparison of obstructions

Theorem 3.4 ([LW23])

Let $\mathcal{X}$ be a smooth alg. k-stack of f.t. that is either DM or Zariski-locally quo. of k-var. by a linear k-group. Then

$$\mathcal{X}(\mathbf{A}_k)^{\mathrm{Br}} = \mathcal{X}(\mathbf{A}_k)^{\text{2-desc}}.$$

Theorem 3.5 ([LW23])

Let $\mathcal{X}$ be an alg. k-stack. Then

$$\mathcal{X}(\mathbf{A}_k)^{\mathrm{Br}} \subseteq \mathcal{X}(\mathbf{A}_k)^{\mathrm{conn}}.$$

- Extend Harari [Har02].

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Comparison of obstructions (cont.)

Theorem 3.6 ([WL24])

Let $\mathcal{X}$ be an algebraic k-stack. Then we have

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- Extend [Sko09, CDX19].

Comparison of obstructions (cont.)

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Descent for Brauer-Manin set along a torsor

- Let $\mathcal{X} = [Y/G]$ where Y is a smooth geometrically integral k-var. and G a conn. linear k-group acting on Y.
- Let $f : Y \to \mathcal{X}$ be the canonic map making Y a G-torosr over $\mathcal{X}$.
- One also defines invariant Brauer subgroup (as in Cao [Cao18]) to be

$$\mathrm{Br}_G\,\mathcal{X} = \{b \in \mathrm{Br}\,\mathcal{X} \mid \rho^* b - p_2^* b \in p_1^*\,\mathrm{Br}\,G\}$$

Theorem 3.7 ([LW23])

Have

$$\mathcal{X}(\mathbf{A}_k)^{\mathrm{Br}} = \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y_\sigma(\mathbf{A}_k)^{\mathrm{Br}_{G_\sigma}(Y_\sigma)}).$$

- Key is to modify Sansuc exact seq. (Thm. 3.1) using Br_G.
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Descent for Brauer-Manin set along a torsor

- Let $\mathcal{X} = [Y/G]$ where Y is a smooth geometrically integral k-var. and G a conn. linear k-group acting on Y.
- Let $f : Y \to \mathcal{X}$ be the canonic map making Y a G-torosr over $\mathcal{X}$.
- One also defines invariant Brauer subgroup (as in Cao [Cao18]) to be

$$\mathrm{Br}_G\,\mathcal{X} = \{b \in \mathrm{Br}\,\mathcal{X} \mid \rho^* b - p_2^* b \in p_1^*\,\mathrm{Br}\,G\}$$

Theorem 3.7 ([LW23])

Have

$$\mathcal{X}(\mathbf{A}_k)^{\mathrm{Br}} = \bigcup_{\sigma \in H^1(k,G)} f_\sigma(Y_\sigma(\mathbf{A}_k)^{\mathrm{Br}_{G_\sigma}(Y_\sigma)}).$$

- Key is to modify Sansuc exact seq. (Thm. 3.1) using Br_G.
- This extends Cao [Cao18].

Outline

Known results

- Let k be a number field.
- The product preservation property of Brauer-Manin set was first established by Skorobogatov and Zarhin [SZ14] for smooth geo. int. projective k-vars.,
- and later by me [Lv20] for open ones.

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Product of alg. stacks

Theorem 3.8 ([LW23])

The functor

$$-(\mathbf{A}_k)^{\mathrm{Br}} : (\mathcal{C}\mathrm{hp}_1)_{/k} \to \mathcal{S}\mathrm{et}$$

preserves finite product, where $(\mathcal{C}\mathrm{hp}_1)_{/k} \subset \mathcal{C}\mathrm{hp}_{/k}$ is the full sub-2-category spanned by smooth alg. k-stacks of f.t

- admitting separated and geo. int. atlases X s.t. $X(\mathbf{A}_k)^{\mathrm{B}} \neq \emptyset$,
- and is either DM or Zariski-locally quos. of k-var. by linear k-groups.

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Corollary 3.9 ([LW23])

If $\mathcal{X}$ and $\mathcal{Y}$ are stacks quos. of smooth geo. int. k-varieties by conn. linear k-groups. Then

$$\mathcal{X}(\mathbf{A}_k)^{\mathrm{Br}} \times \mathcal{Y}(\mathbf{A}_k)^{\mathrm{Br}} \xrightarrow{\sim} (\mathcal{X} \times_k \mathcal{Y})(\mathbf{A}_k)^{\mathrm{Br}}$$

Outline

Motivation

- Recall in Theorem 3.5, how to show the converse inclusion ?
- Want a way both carrying obstructions and satisfying descent.
- A formalism of cohomological descent encoding adèlic points and obstructions.
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Main result

Theorem 4.1 ([Lv24])

Let $\Lambda \in \mathcal{R}\mathrm{ing}_{\square\text{-tor}}$. Then there is a functor

$$\mathcal{C}^{\mathrm{obs}} : \mathrm{N}(\mathcal{C}\mathrm{hp}^{\mathrm{lft},\,\square}_{/S})^{op} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$$

such that for any $f : X_0 \to X_{-1}$ being a smooth surjective map in $\mathcal{C}\mathrm{hp}^{\mathrm{lft},\,\square}_{/S}$ and $X_\bullet : \mathrm{N}(\mathbf{\Delta}_+)^{op} \to \mathcal{C}\mathrm{hp}^{\mathrm{lft},\,\square}_{/S}$ its Čech nerve, the map is an equivalence

$$\mathcal{C}^{\mathrm{obs}}(X_{-1}) \to \varprojlim_{n\in\mathbf{\Delta}} \mathcal{C}^{\mathrm{obs}}(X_n).$$

- The presentable ∞-category $\mathcal{C}^{\mathrm{obs}}(X)$ "contained in" $\mathrm{Fun}(\mathcal{D}(A), \mathcal{D}(X))$ is the "smallest" one containing all x_*, $x \in X(A)^{\mathrm{obs}}$, in a functorial way.

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Outline

Open problems

- For $X \in \mathcal{C}\mathrm{hp}_{/k}$,
- do we have

$$X(\mathbf{A}_k)^{\text{ét,Br}} = X(\mathbf{A}_k)^{\text{desc}} = X(\mathbf{A}_k)^{\text{desc,desc}} \ ?$$

(it is true for varieties)

- Is $X(\mathbf{A}_k)^{\text{2-desc,desc}}$ really smaller than $X(\mathbf{A}_k)^{\text{desc}}$?

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Thanks for your attention.

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