The Brauer–Manin obstruction on algebraic stacks

Joint w/ Han Wu

§1. Generalities

1.1 Points Let $\mathcal{X}$ and $\mathcal{T}$ be stacks over S.

$\mathcal{X}(\mathcal{T}) := \mathrm{Hom}_S(\mathcal{T}, \mathcal{X})/_{\cong}$ — $\mathcal{T}$-points of $\mathcal{X}$.

$S = \mathrm{spec}\, k$

In particular $\mathcal{X}(\mathbb{A}_k)$ — adèlic pts

$\mathcal{X}(k)$ — rat. pts.

$\mathcal{X}(O_k)$ — int. pts.

- sch as sheaf
- alg. space
- alg. stacks DM GS, Art 70 Clp /S
- quo. stacks

1.2 Caution ⚠ Although if $\mathcal{X}$ is rep. by a scheme, this notion coincides with classical one. But

$\underline{\mathcal{X}(k) \longrightarrow \mathcal{X}(\mathbb{A}_k)}$ is NOT ness. inj.

Eg. Let G be a sm affine k-gp.

$$\ker(BG(k) \to BG(\mathbb{A}_k)) = \ker(H^1(k, G) \to H^1_{fppf}(\mathbb{A}_k, G))$$

$$= Ш^1(G/k)$$

1.3 Cohomological obstruction

$\tau \in \{ét, fppf\}$. $\mathcal{G} \in \mathrm{Grp}(S_\tau)$

$F = H^i(-, \mathcal{G}) : \underline{\mathrm{Sh}(S_\tau}, \mathrm{Gpd}) \longrightarrow \mathrm{Set}$.

is a 2-functor (2-iso mapped to identity)

$\Downarrow s \xrightarrow{F} \|$

$\rightsquigarrow \mathcal{X}(\mathbb{A}_k)^A \quad (A \in F(\mathcal{X}))$

$x \in \mathcal{X}(\mathbb{A}_k)$

$F(\mathcal{X})$

$\downarrow x^*$

$F(k) \searrow F(\mathbb{A}_k)$

$\mathcal{X}(\mathbb{A}_k)^F$ are well def. and

$\mathcal{X}(k) \longrightarrow \mathcal{X}(\mathbb{A}_k)^F \subseteq \mathcal{X}(\mathbb{A}_k)^A \subseteq \mathcal{X}(\mathbb{A}_k)$

In particular. we have $\mathcal{X}(\mathbb{A}_k)^{Br}$, $\mathcal{X}(\mathbb{A}_k)^{desc}$

$$\begin{tikzcd}
\mathcal{X}(k) \arrow[r, dashed] \arrow[dr] & \square \arrow[r, "i"] \arrow[d, "\lrcorner"] & \mathcal{X}(\mathbb{A}_k) \arrow[d, "*"] \\
& \mathrm{Map}(F(\mathcal{X}), F(k)) \arrow[r] & \mathrm{Map}(F(\mathcal{X}), F(\mathbb{A}_k))
\end{tikzcd}$$

$\mathrm{im}(i)$

§2. A stacky curve violating Local-global principle for int. pts

2.1. Def (stacky curve) . . .

2.2 Def (Genus) - - . . .

~~**2.3** If $g(X) < \frac{1}{2}$, k - field [BP22]~~

~~(A+) Local-global principle holds and [Chr20] satisfies strong app.~~ Thus looking for $g(X) = \frac{1}{2}$ [BP22] counter-example for $k = \mathbb{Q}$

<u>2.4</u>. "<u>Thm</u>" (Wu–L 22) k # field

$\exists\,(p,q)$ s.t. stacky curve $\mathcal{X}_{(p,q)}$ (of genus $\frac{1}{2}$) violating local-global principle for int. pts.

$$\mathcal{Y}_{(p,q)} := \mathrm{Proj}\left(\mathcal{O}_K[x,y,z]/(z^2 - px^2 - qy^2)\right)$$

$\mu_2 \circlearrowright$ $(x:y:z) \longmapsto (x:y:\lambda z)$

$\lambda \in$

Then $\mathcal{X}_{(p,q)} = [\,\mathcal{Y}_{(p,q)}/\mu_2\,]$.

<u>Motivation</u>, [Bhargava–Poonen 22] stacky $<\frac{1}{2}$, HP ✓ $/\mathcal{O}_K$

$=\frac{1}{2}$ counter $/\mathbb{Z}$

[Wu–L. 23] $\geq\frac{1}{2}$ $/\mathcal{O}_K$

[Christensen 20] ~~top~~ ology, $<\frac{1}{2}$ $/\mathbb{Q}_K$ SA ✓

[Santens 22] stacky / global w/ fin ab. fund. gp SA–BM ✓

<u>§3. Descent by gerbes</u>

<u>3.1</u>. • Recall that descent by torsors

$$X(\mathbb{A}_k)^f = \bigcup_{\sigma \in H^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k))$$

for any $[\,f : Y \to X\,] \in H^1(X,G)$

• We already know [BRAPS 5.5] H^2 classifies gerbes.

3.2 "Prop" (L 21) (descent by gerbe)

consider the cat of stacks over k

$\mathrm{Shv}(k_{fppf}, \mathrm{Gpd})$. $\tau \in \{fppf, \acute{e}t\}$.

For any $\mathcal{G} \in \underline{Ab}(k_\tau)$ and $[f : \mathcal{Y} \to \mathcal{X}] \in H_\tau^2(\mathcal{X}, \mathcal{G})$. we have

$$\mathcal{X}(\mathbb{A}_k)^f = \bigcup_{\sigma \in H_\tau^2(k, \mathcal{G})} f^\sigma(\mathcal{Y}^\sigma(\mathbb{A}_k))$$

A gerbe is a stack that is loc. nonempty & loc. conn.
(bound by some lien (abelian ones) 被何营生…

3.3 Def (Torsors over algebraic stacks)

$\mathcal{X} \in \mathrm{Chp}/S$. $\mathcal{G} \in \mathrm{Shv}(\mathcal{X}_{fppf})$

A $\mathcal{G}$-torsor over $\mathcal{X}_{fppf}$ is a sheaf

$\mathcal{G} \curvearrowright \mathcal{Y} \in \mathrm{Shv}(\mathcal{X}_{fppf})$ s.t. $\mathcal{G} \times \mathcal{Y} \cong \mathcal{Y} \times \mathcal{Y}$

Denoted by $\mathcal{Y} \xrightarrow{\mathcal{G}} \mathcal{X}_\tau \in \mathrm{Tors}(\mathcal{X}_\tau, \mathcal{G})$ gpd

$$\mathrm{Tors}(\mathcal{X}_\tau, \mathcal{G})/_{\cong} \xrightarrow{\sim} \check{H}^1_{fppf}(\mathcal{X}, \mathcal{G})$$

if $\mathcal{G}$ is ab $\downarrow S$

$$H^1_{fppf}(\mathcal{X}, \mathcal{G})$$

3.4 Lemma. G an S-gp sch. $\mathfrak{Y} \in \mathrm{Tors}(X_{fppf}, G)$

then $\mathfrak{Y} \in \mathrm{Chp}/S$. $\quad Y \xrightarrow{G} X \iff X \cong [Y/G]$

3.5 Construction.

$$X(\mathbb{A}_k)^{2\text{-}desc,\ desc} = \bigcap_{\substack{G\ \text{conn} \\ f : Y \to X \in \mathrm{Gerb}(X, G)}} \bigcup_{\sigma \in H^2(k, G)} f^{\sigma}\left(Y^{\sigma}(\mathbb{A}_k)^{\overset{\text{well-def}}{desc}}\right)$$

$$\subseteq X(\mathbb{A}_k)^{desc}.$$

Q. · conter-example for $\subsetneq$?

· or proof for $=$?

§4. More on B-M of

4.1 "thm" (L–Wu 22) (Semisuc exact seq for quotient stacks). Let X/k var. k char $= 0$. G conn. k-gp. $\curvearrowright X$.

$\mathcal{Y} = [X/G]$ $\mathcal{U} := \mathbb{G}_m/k^\times \in PSh(\mathcal{Y})$ where
i.e. $f: X \xrightarrow{G} \mathcal{Y}$ torsor
$k^\times$ is const. then we have exact seq

$$0 \to \mathcal{U}\mathcal{Y} \to \mathcal{U}X \to \mathcal{U}G \to \operatorname{Pic}\mathcal{Y} \to \operatorname{Pic}X \to \operatorname{Pic}G \to$$
$$\operatorname{Br}\mathcal{Y} \xrightarrow{f^*} \operatorname{Br}X \xrightarrow{p^* - p_2^*} \operatorname{Br}(G \times_k X)$$

4.2 Examples In particular for BG,

$\mathcal{U}BG = 0$, $\operatorname{Pic}BG = \mathcal{U}G$ and

$$0 \to \operatorname{Pic}G \to \operatorname{Br}BG \to \operatorname{Br}k \to 0$$
splits.

+4.3 4.4

Colliot-Thélène

4.5 "thm" (Wu–L 22) (Fundamental seq of CT)

$p: X \to k$ alg. stack of ft, k # field.

S k-gp of mul. type $\hat{S}$ Cartier dual

$$KD'(\mathcal{X}) := \mathrm{cone}\left(\mathbb{G}_m[1] \longrightarrow Rp_* \mathbb{G}_m[1] \right).$$

in $D^b(k_{ét})$.

Then we have the fund. ex seq

$$H^1(k,S) \hookrightarrow H^1_{fppf}(\mathcal{X},S) \xrightarrow{\chi} \mathrm{Hom}_{D(k)}(\hat{S}, KD'(\mathcal{X}))$$

$$\longrightarrow H^2(k,S) \to H^2_{fppf}(\mathcal{X},S)$$ where

χ is ~~ea~~ the extended type.

⇒ Two torsors have the same ext. type iff they are iso. up to a twist.

<u>4.6</u> Let $\underline{a} \in H^1(k,\hat{S})$ the diag.

$$\begin{tikzcd}
H^1(\mathcal{X},S) \arrow[r, "\chi"] \arrow[d, "p^*(a)\cup -"'] & \mathrm{Hom}_{D(k)}(\hat{S}, KD'(\mathcal{X})) \arrow[d, "a\cup - = \lambda_* -"] \\
\mathrm{Br}_1 \mathcal{X} \arrow[r, "r"] & H^1(k, KD'(\mathcal{X}))
\end{tikzcd}$$

comm.

For $f : \mathcal{Y} \xrightarrow{S} \mathcal{X} \in \mathrm{Tors}(\mathcal{X}, S)$, define

$\lambda = \chi([f])$ and

$\mathrm{Br}_\lambda \mathcal{X} = r^{-1}(\lambda_*(H^1(k,\hat{S}))) \subseteq \mathrm{Br}_1 \mathcal{X}$.

4.7 "Prop" We have $X(\mathbb{A}_k)^f = X(\mathbb{A}_k)^{\mathrm{Br}\,x}$

4.3 ~~Def~~ (Invariant Br) Follow Ces. $X \xrightarrow{G} Y$ torsor

$$\begin{array}{ccc} G \times_k X & \underset{\rho}{\xrightarrow{p_2}} & X \\ \downarrow p_1 & & \\ G & & \end{array}$$

$\mathrm{Br}_G X := \{ b \in \mathrm{Br}\, X \mid \rho^* b - p_2^* b \in p_1^* \mathrm{Br}\, G \}$

4.4 Cor $X \xrightarrow{G} Y$ sm geo. int k-var, linear conn. k-gp. Have a sub sequence

$$\begin{array}{ccccccccccc} \cdots \to & \mathrm{Br}_1 X & \to & \mathrm{Br}_1 G & \to & \mathrm{Br}\, Y & \to & \mathrm{Br}_G X & \longrightarrow & \mathrm{Br}_e(G) \cong \mathrm{Br}_a G \\ & \| & & \| & & \| & & \downarrow & \boxed{} & \downarrow p_1^* \\ \cdots \to & \mathrm{Br}_1 X & \to & \mathrm{Br}_1 G & \to & \mathrm{Br}\, Y & \xrightarrow{f^*} & \mathrm{Br}\, X & \xrightarrow{\rho^* - p_2^*} & \mathrm{Br}\,(G \times X) \end{array}$$

where $\mathrm{Br}_e G = \ker(\mathrm{Br}_1 G \xrightarrow{e_G^*} \mathrm{Br}\, k)$

4.8 "Cor" (L.-Wu) $X \overset{\text{Zar-locally}}{=} \bigcup\limits_{i \geq 1} [X_i / G_i]$ sm k-var, linear k-gp

↳ need this!

counter-example BE

then $\mathrm{Br}\, X$ is torsion.

4.9 Rmk. For reg. Noe. DM stack X.
$\mathrm{Br}\, X$ is also torsion (Antieau-Meier[20])

4.9" Thm" (L.-Wu) $f: Y \xrightarrow{G} X$ $G \leftarrow$ conn. linear lagr. torsor

Y sm. geo. int for var

Then $X(\mathbb{A}_k)^{Br} = \bigcup_{\sigma \in H^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k)^{Br_G \sigma(Y^\sigma)})$

4.10 • Descent along a torsor for BM set ✓

• Product preservation ?

- Proj sm var. Skorobogatov - Zarhin '14

- sm geo. int var. L. '20 Cao's tech

- algebraic stacks ?

4.11^ Thm" (L.-Wu) The functor $-(\mathbb{A}_k)^{Br}$: $\mathrm{Chp}_{1/k} \to \mathrm{Set}$ preserves fin. prod, where $\mathrm{Chp}_{1/k} \subset \mathrm{Chp}/k$ full subcat spanned by sm. alg. k-stack of f.t.

• admitting sep. geo. int atlas X s.t. $X(\mathbb{A}_k)^{Br_1} \neq \emptyset$

• DM or Zar.-loc. quo of k-var by lin. lagp.

Key ingredient of proof: • torsionness of Br, so can use n

• Existence of univ torsor of n-torsion. $H^*(-,\mu_n)$

$X(\mathbb{A}_k)^{Br_1} \neq \emptyset \Rightarrow \mathcal{X}: H^1_{fppf}(\mathcal{X}, S) \twoheadrightarrow \mathrm{Hom}_{D(k)}(\hat{S}, KD'(\mathcal{X}))$

• Künneth formula for $H^i(-,\mu_n)$, $i=1,2$. $/\bar{k}$

- Künneth for stack: $Rp_* K \boxtimes^L_\Lambda Rq_* L \xrightarrow{\sim} R(p\times q)_*(K \boxtimes^L_\Lambda L)$ coh. desc

- sm bc $p^* Rf_* \xrightarrow{\sim} Rg_* q^*$

Cor "Cor" For stack quo of (Liu–Zheng '17) sm geo. int. k-var by conn. linear k-gp. Hence $\mathcal{X}(\mathbb{A}_k)^{Br} \times \mathcal{Y}(\mathbb{A}_k)^{Br} \xrightarrow{\sim} (\mathcal{X}\times\mathcal{Y})(\mathbb{A}_k)^{Br}$

- Descent along a torsor $\mathcal{X}(\mathbb{A}_k)^{Br} = \bigcup X(\mathbb{A}_k)^{Br_G}$
- $Br_1 \subseteq Br_G$