

The Brauer-Mann obstruction on algebraic stacks

Points w/ Hom Wn

§1. Generalities

1.1. Points Let X and T be stacks over S .

$$X(T) := \text{Hom}_S(T, X) / \cong \longrightarrow T\text{-points of } X.$$

In particular $S = \text{Spec } k$

$X(\mathbb{A}_k)$ — adelic pts

$X(k)$ — red. pts.

$X(\mathcal{O}_k)$ — int. pts.

• sch as sheaf

• alg. space

• alg. stack

• quod. stack

DMG, Art 70

Chp 15

1.2. Caution

Although if X is rep.

by a scheme, this notion coincides with classical one. But

$X(k) \longrightarrow X(\mathbb{A}_k)$ is NOT necessarily inj.

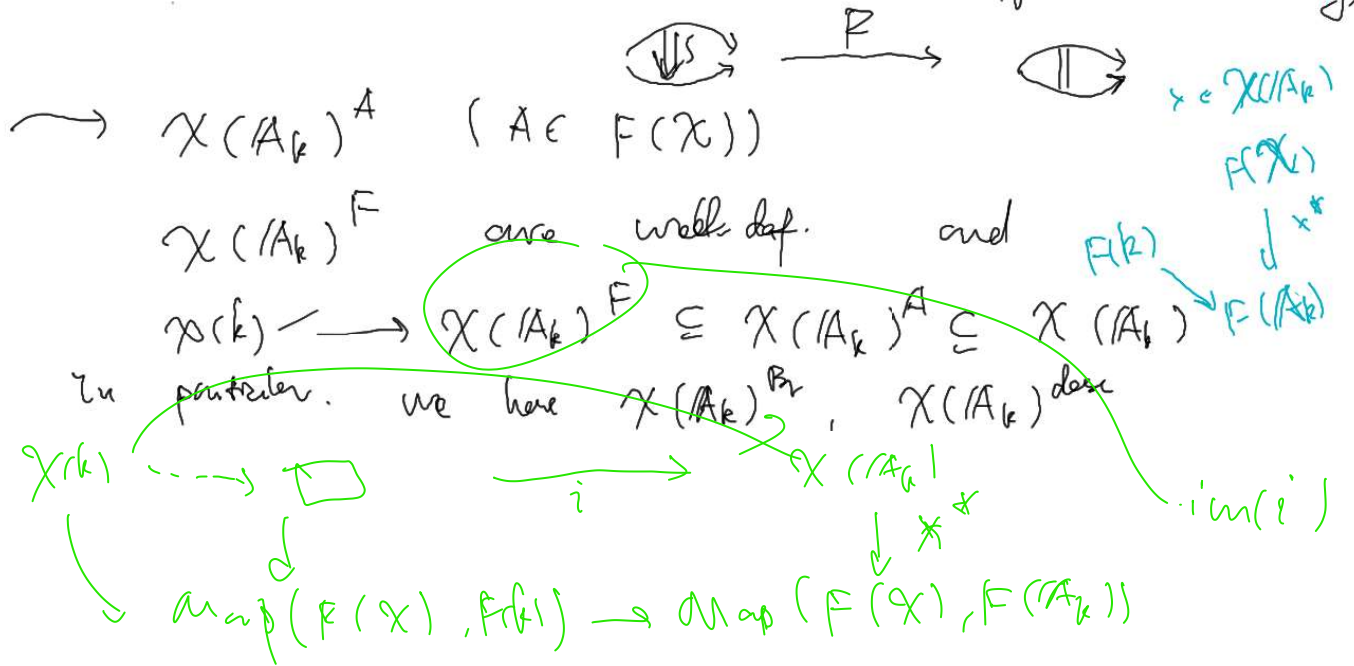
Ex. Let G be a sm affine k -gp.

$$\ker(BG(k) \rightarrow BG(\mathbb{A}_k)) = \ker(H^1(k, G) \rightarrow H^1_{\text{ét}}(\mathbb{A}_k, G))$$

$$= H^1(G/k)$$

1.3 Cohomological obstruction.

$\tau \in \{\text{Set}, \text{FPP}\}$. $\mathcal{G} \in \text{Cmp}(S_\tau)$
 $F = H^1(-, \mathcal{G}) : \text{Shv}(S_\tau, \text{Cpd}) \rightarrow \text{Set}$.
 is a 2-functor (2-iso mapped to identity)



§2. A stacky curve violating local-global principle for int. pts.

2.1. Def (stacky curve) ...

2.2. Def (Genus) - - - -

2.3. If $g(X) < \frac{1}{2}$, k -field [BP22]

(A+) local-global principle holds and [Chr20] satisfies strong app. Thus looking for $g(X) = \frac{1}{2}$ [BP22] counterexample for $k = \mathbb{Q}$

2.4. "thin" (Wu-L 22) $k \neq \text{field}$

$\exists (p, q)$ s.t. stacky curve $X_{(p, q)}$ (of genus $\frac{1}{2}$) violating local-global principle for int. pts.

$$Y_{(p, q)} := \text{Proj}(O_K[x, y, z] / (z^2 - px^2 - qy^2))$$

$$\mu_2 \curvearrowright (x: y: z) \mapsto (x: y: \lambda z)$$

$\lambda \in$ then $X_{(p, q)} = [Y_{(p, q)} / \mu_2]$.

Motivation, (Bhargava-Poonen 22) stacky $< \frac{1}{2}$, HP \vee O_K
 (Wu-L 23) $\mathbb{Z}^1 / O_K = \frac{1}{2}$ center / \mathbb{Z}

[Christensen 20] topology, $< \frac{1}{2}$ O_K SA \vee

[Santens 22] stacky $\S 3$. Descent by gerbes / global w/ fin ab. fund. gr SA-BM \checkmark

3.1. • Recall that descent by torsors

$$X(A_k)^f = \bigcup_{\sigma \in H^1(k, G)} f^\sigma(Y^\sigma(A_k))$$

for any $[f: Y \rightarrow X] \in H^1(X, G)$

• We already know [BRAPS 5.5].

H^2 classifies gerbes.

3.2 "Prop" (L 21) (descent by gerbe)

consider the cat of stacks over k .

$$\text{Shv}(k_{\text{fppf}}, \text{Epd}), \quad \tau \in \{\text{fppf}, \text{ét}\}$$

For any $\mathcal{G} \in \underline{\text{Ab}}(k_\tau)$ and $[f: Y \rightarrow X]$

$\in H_\tau^2(X, \mathcal{G})$ we have

$$X(\mathbb{A}_k)^f = \bigcup_{\sigma \in H_\tau^2(k, \mathcal{G})} f^\sigma(Y^\sigma(\mathbb{A}_k))$$

A gerbe is a stack that is loc. nonempty & loc. comm.
(bound by some lieu) (abelian ansatz) 被同构性...

3.3 Def (Torsors over algebraic stacks)

$$X \in \text{Epd}/S, \quad \mathcal{G} \in \text{Shv}(X_{\text{fppf}})$$

A \mathcal{G} -torsor over X_{fppf} is a sheaf

$$\mathcal{G} \hookrightarrow \mathcal{Y} \in \text{Shv}(X_{\text{fppf}}) \quad \text{s.t.} \quad \mathcal{G} \times \mathcal{Y} \cong \mathcal{Y} \times \mathcal{Y}$$

denoted by $\mathcal{Y} \xrightarrow{\mathcal{G}} X_\tau \in \text{Tors}(X_\tau, \mathcal{G})$ and

$$\text{Tors}(X_\tau, \mathcal{G}) / \cong \xrightarrow{\sim} \check{H}_{\text{fppf}}^1(X, \mathcal{G})$$

$$\text{if } \mathcal{G} \text{ is ab.} \quad \downarrow S$$

$$H_{\text{fppf}}^1(X, \mathcal{G})$$

3.4 Lemma - G an S -gp sch. $\sigma_f \in \text{Tor}(X_{\text{form}}, G)$

then $\sigma_f \in \mathbb{C}h_p/S$.

$$Y \xrightarrow{G} X \iff X \cong [Y/G]$$

3.5 Construction

$$X(A_k)^{\text{2-desc, desc}} = \bigcap_{G \text{ conn}} \bigcup_{\sigma \in H^2(k, G)} f^\sigma(Y^\sigma(A_k)^{\text{disc}})$$

\downarrow
well-defined

$f: Y \rightarrow X \in \text{Covh}(X, G)$

$$\subseteq X(A_k)^{\text{disc}}$$

Q

- counter-example for \subseteq ?
- or proof for $=$?

§4. More on B-M of

4.1 "rlm" (L. - Cu 22) (Semisuc exact seq for quotient stacks)

Let X/k var. k char = 0. G conn. k -gp. $\mathcal{Y} \subset X$.

$$\mathcal{Y} = [X/G]. \quad U := \Gamma_{\mathcal{Y}}/k^* \in \text{PSH}(\mathcal{Y}) \text{ where}$$

i.e. $f: X \xrightarrow{G} \mathcal{Y}$ torsor k^* is const. then we have exact seq

$$0 \rightarrow U\mathcal{Y} \rightarrow UX \rightarrow UG \rightarrow \text{Pic } \mathcal{Y} \rightarrow \text{Pic } X \rightarrow \text{Pic } G \rightarrow \text{Br } \mathcal{Y} \xrightarrow{f^*} \text{Br } X \xrightarrow{P^* - P_2^*} \text{Br}(G \times_k X)$$

4.2 Example Zn parictalon for $\text{Br } G$

$$U\text{Br } G = 0, \quad \text{Pic } \text{Br } G = UG \text{ and}$$

$$0 \rightarrow \text{Pic } G \rightarrow \text{Br } \text{Br } G \rightarrow \text{Br } k \rightarrow 0$$

splits.

+ 4.3 4.4

Colliot-Thélène

4.5 "rlm" (Wu-L 22) (Fundamental seq of CT)

$p: X \rightarrow k$ alg. stack of ft, $k \neq \mathbb{F}$ field.

S k -gp of unal. type. \hat{S} Cartier dual.

$$KD'(\mathcal{X}) := \text{cone} \left(\mathbb{G}_m[1] \rightarrow R\hat{p}_* \mathbb{G}_m[1] \right) \\ \text{in } D^b(k\hat{S}).$$

Then we have the fund. ex seq.

$$H^i(k, S) \xrightarrow{\sim} H^i_{\text{fppf}}(\mathcal{X}, S) \xrightarrow{\chi} \text{Hom}_{D(k)}(\hat{S}, KD'(\mathcal{X})) \\ \longrightarrow H^2(k, S) \xrightarrow{\sim} H^2_{\text{fppf}}(\mathcal{X}, S) \quad \text{where.}$$

χ is of the extended type.

\Rightarrow Two torsors have the same ext. type iff they are iso. up to a twist.

4.6 Let $\underline{a} \in H^i(k, \hat{S})$ + the diag.

$$\begin{array}{ccc} H^i(\mathcal{X}, S) & \xrightarrow{\chi} & \text{Hom}_{D(k)}(\hat{S}, KD'(\mathcal{X})) \\ \downarrow p^*(a) \cup - & & \downarrow a \cup - = \lambda_* - \\ \text{Br}_i \mathcal{X} & \xrightarrow{r} & H^i(k, KD'(\mathcal{X})) \end{array}$$

commute.

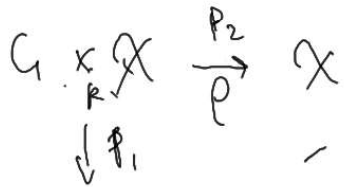
For $f: Y \xrightarrow{S} \mathcal{X} \in \text{Tors}(\mathcal{X}, S)$, define.

$$\lambda = \chi(f) \quad \text{and}$$

$$\text{Br}_\lambda \mathcal{X} = r^{-1}(\lambda_*(H^i(k, \hat{S}))) \subseteq \text{Br}_i \mathcal{X}.$$

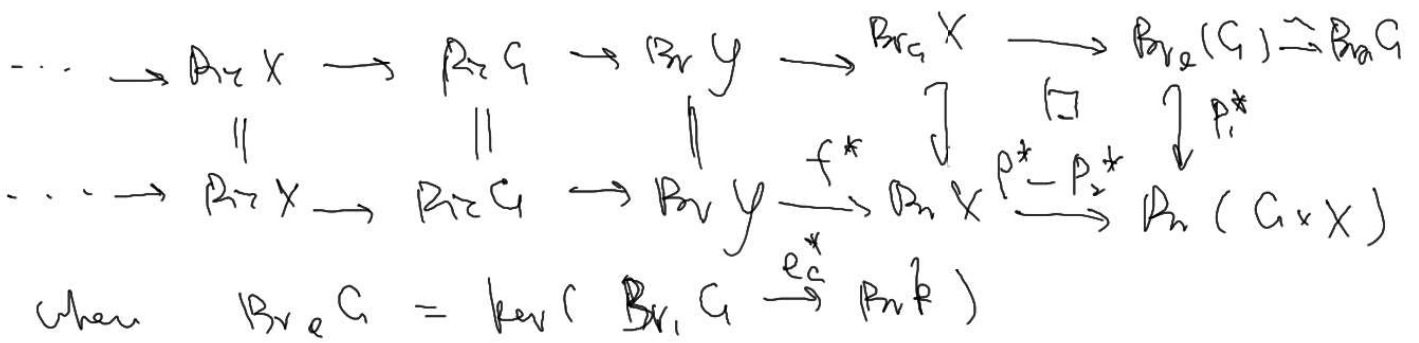
4.7 "Prop" We have $X(A_k)^f = X(A_k)^{Br_X}$

4.3 Def (Invariant Div) Following case, $X \xrightarrow{G} Y$ ^{torsion}



$$Br_G X := \{ b \in Br X \mid p_1^* b - p_2^* b \in p_1^* Br G \}$$

4.4 Cor $X \xrightarrow{G} Y$
 in gen. dim k -var
 linear conn. kept
 hence a sub sequence



4.8 "Cor" (L.-Wu) $X \cong \bigcup_{i=1}^n [X_i / G_i]$
 2-locally
 linear kept
 need this!
 counter-example
 BE
 then $Br X$ is torsion
 in k -var

4.9 Rank. For reg. Noe. D.M. stack X ,
 $Br X$ is also torsion. (Antieau - Meier)

4.9 "thm" (L. - Wu) $f: Y \xrightarrow{G \leftarrow \text{conv. linear Lgr. torsor}} X$
 \uparrow $\text{sm. geo. int. var.}$

Then $X(A_k)^{Br} = \bigcup_{\sigma \in H^1(k, G)} f^\sigma(Y^\sigma(A_k)^{Br})$

- 4.10
- Present along a torsor for BM Set ✓
 - Product preservation?
 - Proj sm var, Skovfogator-Zarkhin 14
 - sm geo. int var. L. 20 Cao's tech
 - algebraic stacks?

4.11 "thm" (L. - Wu) The functor $-(A_k)^{Br}: \text{Chp}/k \rightarrow \text{Set}$
 preserves fin. prod; where $\text{Chp}/k \subset \text{Chp}/k$ full
 subcat spanned by sm. alg. k-stacks of f.t.

- admitting sep. geo. int atlas X c.f. $X(A_k)^{Br} \neq \emptyset$
- DM or Zar-Loc. geo of k-var by linear Lgr.

- Key ingredient of proof:
- Tor'sness of Br can use $H^k(-, \mu_n)$
 - Existence of univ torsor of n -torsion
 $X(A_k)^{Br} \neq \emptyset \Rightarrow X: H_{\text{f.t.}}^1(X, S) \rightarrow H_{\text{f.t.}}^1(\hat{S}, \text{Ker}(S))$
 - Kato's formula for $H^i(-, \mu_n)$, $i=1, 2$ / \mathbb{F}_k

- Künneth for stacks: $Rp_* K \otimes_A^L Rq_* L$
 $\xrightarrow{\sim} R(p \times q)_* (K \otimes_A^L L)$ coh. desc

- Sum bc $p^* R f_* \xrightarrow{\sim} R g_* q^*$

6.12 "Cor"

for stack quo of Liu-Zhang 17

sum geo. nit. k -var by coun. linear
 k -gp. where $X(A_k)^{P_{n_1}} \times Y(A_k)^{P_{n_2}} \cong (X \times Y)(A_k)^{P_{n_1} \times P_{n_2}}$

- Present along a torson $X(A_k)^{P_{n_1}} = \cup X(A_k)^{P_{n_2}}$
- $P_{n_1} \subseteq P_{n_2}$