

The Brauer-Mann obstruction on algebraic stacks

§1. Generalities

1.1 Points Let X and T be stacks over S .

$$X(T) := \text{Hom}_S(T, X)_{\text{iso}} \quad \text{---} \quad T\text{-points of } X.$$

$S = \text{spec } k$

In particular

$$X(A_k) \quad \text{---} \quad \text{adèlic pts}$$

$$X(k) \quad \text{---} \quad \text{rat. pts.}$$

$$X(O_k) \quad \text{---} \quad \text{int. pts.}$$

1.2

Caution



Although if X is rep.

by a scheme, this notion coincides with classical one. But

$X(k) \longrightarrow X(\mathbb{A}_k)$ is NOT necessarily inj.

Def. Let G be an sm affine k -gp.

$$\ker (BG(k) \longrightarrow BG(\mathbb{A}_k)) = \ker (H'(k, G) \longrightarrow H'_{\text{fppf}}(\mathbb{A}_k, G))$$

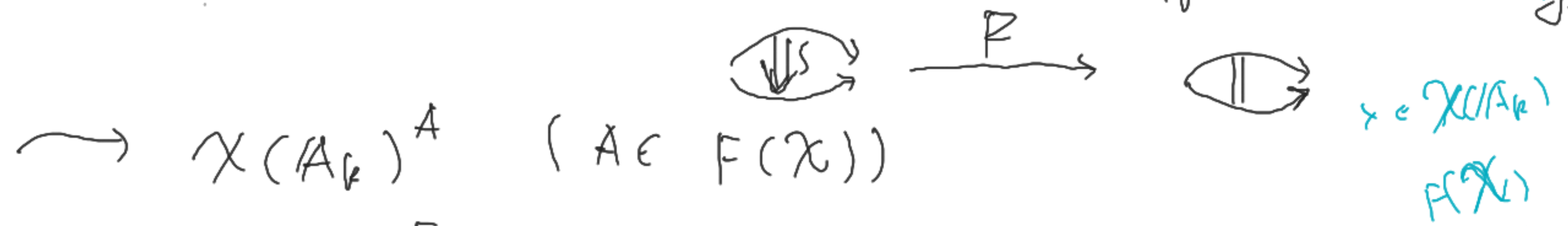
$$= III'(G/k)$$

1.3 Cohomological obstruction.

$\tau \in \{ \text{ét}, \text{fppf} \} . \quad \mathcal{G} \in \text{Grp}(S_\tau)$

$F = H^i(-, \mathcal{G}) : \text{Shv}(S_\tau, \text{Grp}) \rightarrow \text{Set} .$

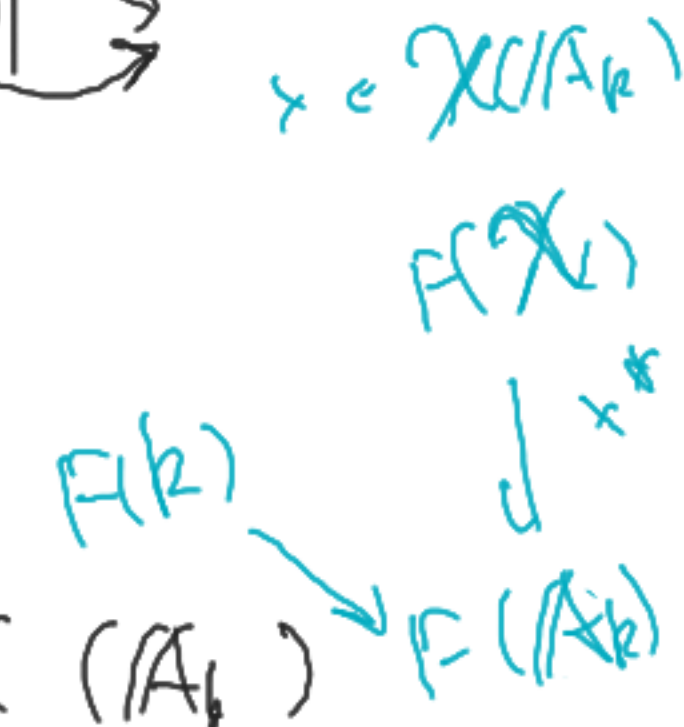
is a τ -functor (\mathbb{Z} -iso mapped to identity)



$X(A_k)^F$ are well def. and

$X(k) \rightarrow X(A_k)^F \subseteq X(A_k)^A \subseteq X(A_k)$

In particular, we have $X(A_k)^{Br}$, $X(A_k)^{disc}$



§ 2. A stably curve violating
Local-global principle for int. pts.

2.1. Def (Stably curve) ...

2.2. Def (Genus) - - -

2.3. If $g(X) < \frac{1}{2}$, k - field. [BP22]

(A+) Local-global principle holds and [Chr20]
satisfies strong app. Thus looking for

$g(X) = \frac{1}{2}$ [BP22] counter-example for $k = \mathbb{Q}$

2.4. "Thm" (Wu-L 22) k # field

$\exists (p, q)$ s.f. stacky curve $X_{(p, q)}$ (of

genus $\frac{1}{2}$) violating local-global

principle for int. pts.

$$Y_{(p, q)} := \text{Proj}(U_K[x, y, z] / (z^2 - px^2 - qy^2))$$

μ_2

$$(x : y : z) \mapsto (x : y : \lambda z)$$

$\lambda \in$

$$\text{Thm } X_{(p, q)} = [Y_{(p, q)} / \mu_2].$$

§ 3. Descent by gerbes

3.1. • Recall that descent by torsors

$$X(A_k)^f = \bigcup_{\sigma \in H^1(k, G)} f^\sigma(Y^\sigma(A_k))$$

for any $[f: Y \rightarrow X] \in H^1(X, G)$

• We already know [BRAPS 5.5].

H^2 classifies gerbes.

Σ 2 "Frag", (L 21) (descent by gerbe)

consider the cat of stacks over k .

$\text{Shv}(k_{\text{fppf}}, \text{Epd})$. $\tau \in \{\text{fppf}, \text{ét}\}$.

For any $\mathcal{G} \in \underline{\text{Ab}}(k_\tau)$ and $[f: Y \rightarrow X]$

$\in H_\tau^2(X, \mathcal{G})$. we have

$$X(\mathbb{A}_k)^f = \bigcup_{\sigma \in H_\tau^2(k, \mathcal{G})} f^\sigma(Y^\sigma(\mathbb{A}_k))$$

A gerbe in a stack that is in loc. nonempty & loc. comm. (found by some lieu. (abelian ones) 被 (to) 5 4...

3.3 Def (Torsors over algebraic stacks)

$X \in \text{Chp/S}$, $\mathcal{G} \in \text{Shv}(X_{\text{fppf}})$

A \mathcal{G} -torsor over X_{fppf} is a sheaf

$\mathcal{G} \hookrightarrow \mathcal{Y} \in \text{Shv}(X_{\text{fppf}})$ s.t. $\mathcal{G} \times \mathcal{Y} \cong \mathcal{Y} \times \mathcal{Y}$

Denoted by $\mathcal{Y} \xrightarrow{\mathcal{G}} X$ $\in \text{Tors}(X, \mathcal{G})$ and

$$\text{Tors}(X, \mathcal{G}) \cong \check{H}_{\text{fppf}}^1(X, \mathcal{G})$$

If \mathcal{G} is ab. \downarrow

$$H_{\text{fppf}}^1(X, \mathcal{G})$$

3.4 Lemma - G an S -gp sch. $\mathcal{O}_Y \in \text{Tor}(X_{\text{form}}, G)$

then $\mathcal{O}_Y \in \mathbb{C}hp/S$.

$$Y \xrightarrow{G} X \iff X \cong [Y/a]$$

3.5

Construction

unhelpful
↓

$$X(A_k)^{\text{2-desc, desc}} = \bigcap_{G \text{ conn}} \bigcup_{\sigma \in H^2(k, G)} f^\sigma \left(Y^\sigma(A_k)^{\text{desc}} \right)$$

$f: Y \rightarrow X \in \text{Covh}(X, G)$

$$\subseteq X(A_k)^{\text{desc}}$$

Q

- counter-example. for \neq ?
- or proof for $=$?

§4 Move on B-M of

4.1 "4.1" (L. - CVu 22) (Semi-loc. exact seq for quotient stacks)

var. k . char = 0. G conn. k -gp. $\mathcal{Y} \subset X$.

$\mathcal{Y} = [X/G]$. $\mathcal{U} := \mathcal{O}_{G^m}/k^x \in \text{PSH}(\mathcal{Y})$ where

i.e. $f: X \xrightarrow{G} \mathcal{Y}$ torsor k^x is const. then we have exact seq

$$0 \rightarrow \mathcal{U} \mathcal{Y} \rightarrow \mathcal{U} X \rightarrow \mathcal{U} G \rightarrow \text{Pic } \mathcal{Y} \rightarrow \text{Pic } X \rightarrow \text{Pic } G \rightarrow$$

$$\text{Br } \mathcal{Y} \xrightarrow{f^*} \text{Br } X \xrightarrow{p_1^* - p_2^*} \text{Br}(G \times_k X)$$

4.2 Example In particular for BrG ,

$$H^1(BG) = 0, \quad Pic BG = H^1(G) \text{ and.}$$

$$0 \rightarrow Pic G \rightarrow Br BG \rightarrow Br k \rightarrow 0$$

+4.3 ← 4.4

splits.

Colliot-Thélène

4.5 "phm" (Wu-L. 22) (Fundamental seq of CT)

$p: X \rightarrow k$ alg. stack of f^* , $k \neq \mathbb{F}$ field.

S k -gp of unal. type. \hat{S} Cartier dual.

$$KD'(\mathcal{X}) := \text{cone}(\mathbb{G}_m[1] \rightarrow \mathbb{R}p_* \mathbb{G}_m[1]).$$

in $D^b(k\text{-mod})$.

Then we have the fund. ex seq.

$$H^1(k, S) \hookrightarrow H^1_{\text{fppf}}(\mathcal{X}, S) \xrightarrow{\mathcal{X}} \text{Hom}_{D(k)}(\hat{S}, KD'(\mathcal{X}))$$

$$\longrightarrow H^2(k, S) \hookrightarrow H^2_{\text{fppf}}(\mathcal{X}, S) \quad \text{where.}$$

\mathcal{X} is the extended type.

\Rightarrow Two torsors have the same ext. type iff they are iso. up to a twist.

4.6

Let $\underline{a} \in H^1(k, \hat{S})$ + the diag.

$$\begin{array}{ccc} H^1(X, S) & \xrightarrow{\chi} & \text{Hom}_{\text{D}(k)}(\hat{S}, \text{K}D'(X)) \\ \downarrow p^*(a) \cup - & & \downarrow a \cup - = \lambda_* - \end{array}$$

$$\text{Br}_1 X \xrightarrow{r} H^1(k, \text{K}D'(X))$$

comm.

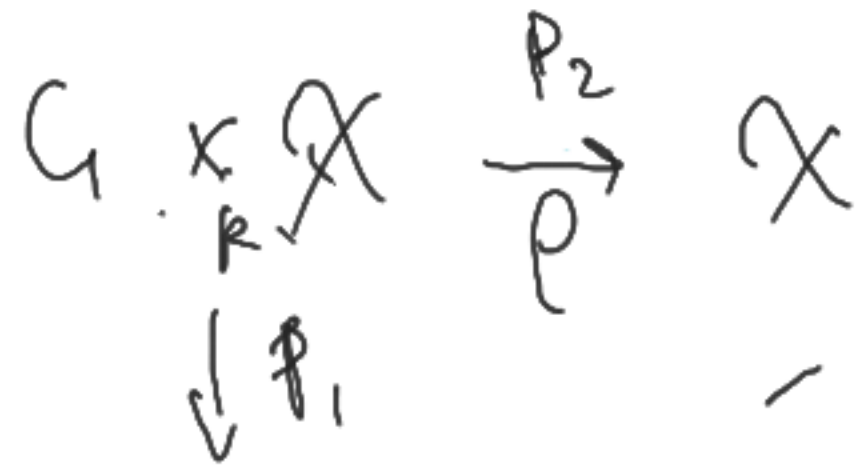
For $f: Y \xrightarrow{S} X \in \text{Tors}(X, S)$, define.

$$\lambda = \chi([f]) \quad \text{and}$$

$$\text{Per}_\lambda X = r^{-1}(\lambda_*(H^1(k, \hat{S}))) \subseteq \text{Per}_1 X.$$

4.7. "Prop" We have $\chi(A_k)^f = \chi(A_k)^{\text{Br}_\lambda}$

4.3 Def (Invariant (Pr)) Following case, $X \xrightarrow{G} Y$



$$\text{Pr}_G X := \{ b \in \text{Pr} X \mid \rho_1^* b - \rho_2^* b \in \rho_1^* B \}$$

4.4 Cor

$X \xrightarrow{G} Y$

can geo. det k -var
 linear conn. keep
 Hence a sub sequence

$$\begin{array}{ccccccccccc}
 \dots & \rightarrow & \mathbb{P}^n X & \rightarrow & \mathbb{P}^n G & \rightarrow & \mathbb{P}^n Y & \rightarrow & \mathbb{P}^n X & \rightarrow & \mathbb{P}^n(G) \xrightarrow{\sim} \mathbb{P}^n G \\
 & & \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\
 \dots & \rightarrow & \mathbb{P}^n X & \rightarrow & \mathbb{P}^n G & \rightarrow & \mathbb{P}^n Y & \xrightarrow{f^*} & \mathbb{P}^n X & \xrightarrow{p_1^* - p_2^*} & \mathbb{P}^n(G \times X) \\
 & & & & & & & & & & \downarrow p_i^*
 \end{array}$$

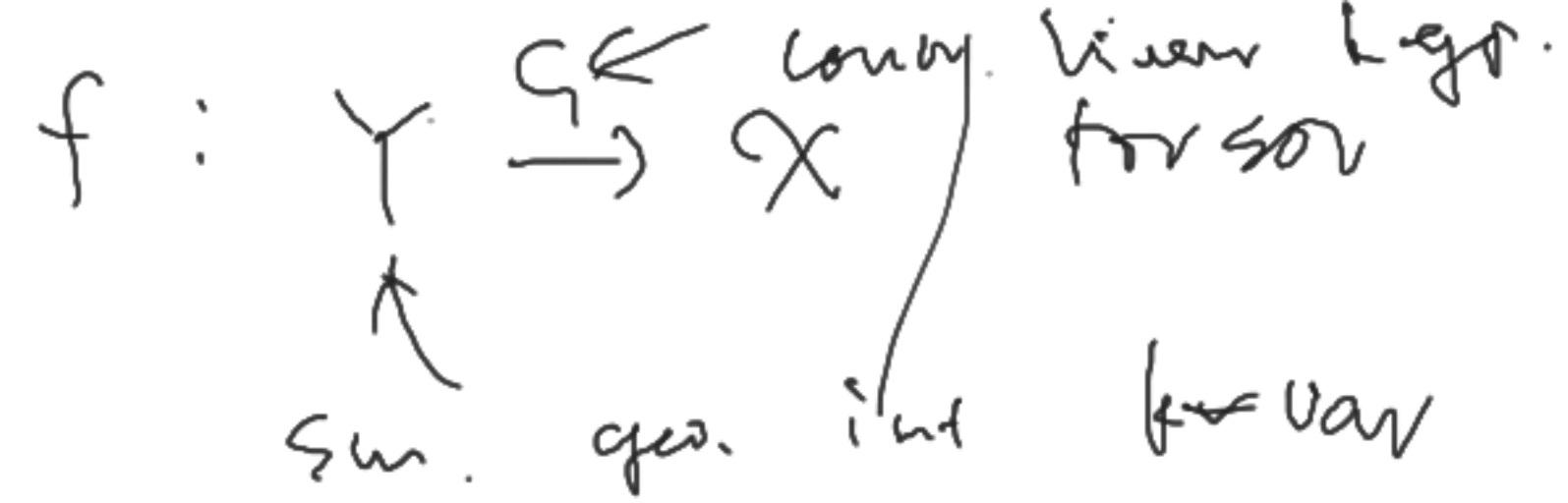
where $\mathbb{P}^n G = \ker(\mathbb{P}^n G \xrightarrow{p_i^*} \mathbb{P}^n k)$

Def "Cov" $(L, W_n) X = \bigcup_{i=1}^n [X_i / G_i]$ given by

then $\mathbb{P}^n X$ is torsion see above

4.9 Rank . For reg. Noe. Dan stack X , $\mathbb{P}^n X$ is also torsion. (Antieau - Meier)

4.9 "thm" (L. - MVu)



Then $X(A_k)^{Br} = \bigcup_{\sigma \in H(k, G)} f^{\sigma} (Y^{\sigma} (A_k)^{Br^{\sigma}} (Y^{\sigma}))$

- 4.10
- o Present along a torsor for BM. Set \checkmark
 - o Product preservation ?
 - o Proj sm var. Sheaf cohomology - Zariski 14
 - o sm geo. int var. L. 20
 - o algebraic stacks ?

4.11 Thm (L. - Wu) The functor $-(A_k)^{Br} : \mathcal{C}h_p/k \rightarrow \text{Set}$

preserves fin. prod; where $\mathcal{C}h_p/k \subset \mathcal{C}h_p/k$ full

subcat spanned by sm. alg. k -stack of f.f.

• admitting sep. geo. int atlas X c.f. $X (A_k)^{Br} \neq \emptyset$

• DM or Zar - Coc. quo of k -var by linear hyp.

Key ingredient of proof: • torsionness of Br can use $H^*(-, \mu_n)$

• Existence of univ torsor of n -torsion.

$X (A_k)^{Br} \neq \emptyset \Rightarrow X : H_{\text{fppf}}^i(X, S) \rightarrow H_{\text{fppf}}^i(\hat{S}, K_0(S))$

• Kato's formula for $H^i(-, \mu_n)$, $i=1, 2$. \mathbb{A}_k

- Künneth for stacks : $R p_* K_X^L \cong R q_* L$

$\xrightarrow{\sim} R(p \times q)_* (K_X^L \otimes L)$ coh. desc

- Sum bc $p^* R f_* \xrightarrow{\sim} R g_* q^*$

4.12 "Cor"

For stack. quo of Liu-Zheng '17

sum gen. with k -var by coun. linear
 k -gp. Hom $X(A_k)^{P_n} \times Y(A_k)^{P_n} \cong (X \times Y)(A_k)^{P_n}$

• Present along a torson $X(A_k)^{P_n} = \bigcup X(A_k)^{P_n, \sigma}$

• $P_n, \sigma \subseteq P_n, \alpha$