The Brauer–Manin obstruction on algebraic stacks

§1. Generalities

1.1. Points Let $\mathcal{X}$ and $\mathcal{T}$ be stacks over S.

$$\mathcal{X}(\mathcal{T}) := \mathrm{Hom}_S(\mathcal{T}, \mathcal{X})/_{\cong} \quad \text{— } \mathcal{T}\text{-points of } \mathcal{X}.$$

$S = \operatorname{spec} k$

In particular

$\mathcal{X}(\mathbb{A}_k)$ — adelic pts

$\mathcal{X}(k)$ — rat. pts.

$\mathcal{X}(\mathcal{O}_k)$ — int. pts.

1.2 Caution Although if X is rep. by a scheme, this notion coincides with classical one. But

$$X(k) \longrightarrow X(\mathbb{A}_k) \text{ is NOT ness. inj.}$$

Eg. Let G be a sm affine k-gp.

$$\ker\left(BG(k) \longrightarrow BG(\mathbb{A}_k)\right) = \ker\left(H^1(k, G) \longrightarrow H^1_{fppf}(\mathbb{A}_k, G)\right)$$

$$= Ш^1(G/k)$$

1.3 Cohomological obstruction

$\tau \in \{\text{ét}, \text{fppf}\}$. $\qquad \mathcal{G} \in \mathrm{Grp}(S_\tau)$

$F = H^i(-, \mathcal{G}) : \mathrm{Sh}(S_\tau, \mathrm{Gpd}) \longrightarrow \mathrm{Set}$.

is a 2-functor (2-iso mapped to identity)

$$\Downarrow s \quad \xrightarrow{F} \quad \|$$

$\longrightarrow \mathcal{X}(\mathbb{A}_k)^A \quad (A \in F(\mathcal{X}))$

$\mathcal{X}(\mathbb{A}_k)^F$ are well def. and

$$\mathcal{X}(k) \longrightarrow \mathcal{X}(\mathbb{A}_k)^F \subseteq \mathcal{X}(\mathbb{A}_k)^A \subseteq \mathcal{X}(\mathbb{A}_k)$$

$x \in \mathcal{X}(\mathbb{A}_k)$

$$\begin{tikzcd} & F(\mathcal{X}) \arrow[d, "x^*"] \\ F(k) \arrow[r] & F(\mathbb{A}_k) \end{tikzcd}$$

In particular. we have $\mathcal{X}(\mathbb{A}_k)^{\mathrm{Br}}$, $\mathcal{X}(\mathbb{A}_k)^{\mathrm{desc}}$

§2. A stacky curve violating local-global principle for int. pts.

2.1. Def (stacky curve) ......

2.2 Def (Genus) ― - . . . .

Rk.3 If $g(X) < \frac{1}{2}$, k – field [BP22]

(A+) local-global principle holds and [Chr20]

satisfying strong app. Thus looking for

$g(X) = \frac{1}{2}$ [BP22] counter-example for $k = \mathbb{Q}$

2.4 "Thm" (Wu-L 22) k # field

$\exists\ (p, q)$ s.t. stacky curve $\mathcal{X}_{(p,q)}$ (of genus $\frac{1}{2}$) violating local-global principle for int. pts.

$$\mathcal{Y}_{(p,q)} := \mathrm{Proj}\left(\mathcal{O}_K[x, y, z] / (z^2 - px^2 - qy^2)\right)$$

$$\mu_2 \curvearrowright \quad (x : y : z) \longmapsto (x : y : \lambda z)$$

$\lambda \in \mu_2$

Then $\mathcal{X}_{(p,q)} := [\mathcal{Y}_{(p,q)} / \mu_2]$.

§ 3. Descent by gerbes

3.1. • Recall that descent by torsors

$$X(\mathbb{A}_k)^f = \bigcup_{\sigma \in H^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k))$$

for any $[f : Y \to X] \in H^1(X, G)$

• We already know [BRAPS 5.5]

H^2 classifies gerbes.

§2 "Prop" (L 21) (descent by gerbe)

consider the cat of stacks over k

$Shv(k_{fppf}, Gpd)$. $\tau \in \{fppf, ét\}$.

For any $\mathcal{G} \in \underline{Ab}(k_\tau)$ and $[f : \mathcal{Y} \to \mathcal{X}]$ $\in H^2_\tau(\mathcal{X}, \mathcal{G})$. we have

$$\mathcal{X}(\mathbb{A}_k)^f = \bigcup_{\sigma \in H^2_{\tau}(k, \mathcal{G})} f^\sigma(\mathcal{Y}^\sigma(\mathbb{A}_k))$$

A gerbe is a stack that is loc. nonempty & loc. conn.
(bound by some lien (abelian ones) 设陌生些…

3.3 Def (Torsors over algebraic stacks)

$\mathcal{X} \in \mathrm{Chp}/S$. $\mathcal{G} \in \mathrm{Shv}(\mathcal{X}_{fppf})$.

A $\mathcal{G}$-torsor over $\mathcal{X}_{fppf}$ is a sheaf

$\mathcal{G} \curvearrowright \mathcal{Y} \in \mathrm{Shv}(\mathcal{X}_{fppf})$ s.th.. $\mathcal{G} \times \mathcal{Y} \xrightarrow{\sim} \mathcal{Y} \times \mathcal{Y}$

Denoted by $\mathcal{Y} \xrightarrow{\mathcal{G}} \mathcal{X}_\tau \in \mathrm{Tors}(\mathcal{X}_\tau, \mathcal{G})$ grpd

$$\mathrm{Tors}(\mathcal{X}_\tau, \mathcal{G})/\simeq \xrightarrow{\sim} \check{H}^1_{fppf}(\mathcal{X}, \mathcal{G})$$

if $\mathcal{G}$ is ab $\quad \downarrow \wr$

$$H^1_{fppf}(\mathcal{X}, \mathcal{G})$$

3.4 Lemma. G an S-gp sch. $\mathcal{Y} \in \mathrm{Tors}(\mathcal{X}_{fppf}, G)$

Then $\mathcal{Y} \in \mathrm{Chp}/S$.

$$\mathcal{Y} \xrightarrow{G} \mathcal{X} \longleftrightarrow \mathcal{X} \cong [\mathcal{Y}/G]$$

3.5 Construction

welldefined

$$X(\mathbb{A}_k)^{2\text{-desc},\,\text{desc}} = \bigcap_{\substack{G \text{ conn} \\ f: Y \to X \in \mathrm{Tors}(X,G)}} \bigcup_{\sigma \in H^2(k,G)} f^{\sigma}\left(Y^{\sigma}(\mathbb{A}_k)^{\text{desc}}\right)$$

$$\subseteq X(\mathbb{A}_k)^{\text{desc}}.$$

Q.

- counter-example for $\subsetneq$?
- or proof for $=$?

§4. More on B-M of .

4.1 "Thm" (L-Wu 22) (Semisuc exact seq for quotient stacks). Let X/k be var. k char $=0$. G conn k-gp. $\curvearrowright X$.

$\mathcal{Y} = [X/G]$. $U := \mathbb{G}_m/k^\times \in PSh(\mathcal{Y})$ where

i.e. $f : X \xrightarrow{G} \mathcal{Y}$ torsor

$k^\times$ is const. Then we have exact seq

$$0 \to U\mathcal{Y} \to UX \to UG \to \operatorname{Pic}\mathcal{Y} \to \operatorname{Pic}X \to \operatorname{Pic}G \to$$
$$\operatorname{Br}\mathcal{Y} \xrightarrow{f^*} \operatorname{Br}X \xrightarrow{p_1^* - p_2^*} \operatorname{Br}(G \times_k X)$$

4.2 Example: In particular for $\mathrm{Br}\,G$,

$U\,BG = 0$, $\mathrm{Pic}\,BG = U\,G$ and

$$0 \to \mathrm{Pic}\,G \longrightarrow \mathrm{Br}\,BG \longrightarrow \mathrm{Br}\,k \longrightarrow 0$$

splits.

+ 4.3 4.4

4.5 "thm" (Wu-L 22) (Fundamental seq of CT) Colliot-Thélène

$p : X \to k$ alg. stack of f.t, k # field

S k-gp of mul type $\hat{S}$ Cartier dual

$$KD'(X) := \mathrm{cone}\left(\mathbb{G}_m[1] \longrightarrow Rp_* \mathbb{G}_m[1]\right).$$

in $D^b(k_{ét})$.

Then we have the fund. ex seq.

$$H^1(k, S) \hookrightarrow H^1_{fppf}(X, S) \xrightarrow{\chi} \mathrm{Hom}_{D(k)}(\hat{S}, KD'(X))$$

$$\longrightarrow H^2(k, S) \to H^2_{fppf}(X, S) \quad \text{where}.$$

χ is the extended type.

$\Rightarrow$ Two torsors have the same ext. type iff they are iso. up to a twist.

<u>4.6</u> Let $\underline{a} \in H^1(k, \hat{S})$ the diag.

$$\begin{tikzcd} H^1(X, S) \arrow[r, "\chi"] \arrow[d, "p^*(a) \cup -"'] & \mathrm{Hom}_{D(k)}(\hat{S}, KD'(X)) \arrow[d, "a \cup - = \lambda_* -"] \\ \mathrm{Br}_1 X \arrow[r, "r"] & H^1(k, KD'(X)) \end{tikzcd}$$

commc.

For $f : Y \xrightarrow{S} X \in \mathrm{Tors}(X, S)$, define

$\lambda = \chi([f])$ and

$\mathrm{Br}_\lambda X = r^{-1}(\lambda_*(H^1(k, \hat{S}))) \subseteq \mathrm{Br}_1 X$.

4.7 "Prop" We have $X(\mathbb{A}_k)^f = X(\mathbb{A}_k)^{\mathrm{Br}_X}$

4.3 ~~Def~~ (Invariant Br) Following Ces. $X \xrightarrow{G} Y$ torsor

$$\begin{array}{ccc} G \times_k X & \xrightarrow[\rho]{p_2} & X \\ \downarrow p_1 & & \\ G & & \end{array}$$

$$\mathrm{Br}_G X := \{ b \in \mathrm{Br}\, X \mid \rho^* b - p_2^* b \in p_1^* \mathrm{Br}\, G \}$$

4.4 Cor $X \xrightarrow{G} Y$

sm geo. int k-var

linear conn. k-gp

Hence a sub sequence

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Br}_2 X \arrow[r] \arrow[d, equal] & \mathrm{Br}_2 G \arrow[r] \arrow[d, equal] & \mathrm{Br}\, Y \arrow[r] \arrow[d, equal] & \mathrm{Br}_G X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Br}_e(G) \cong \mathrm{Br}_a G \arrow[d, "p_1^*"] \\
\cdots \arrow[r] & \mathrm{Br}_2 X \arrow[r] & \mathrm{Br}_2 G \arrow[r] & \mathrm{Br}\, Y \arrow[r, "f^*"] & \mathrm{Br}\, X \arrow[r, "p_1^* - p_2^*"] & \mathrm{Br}\,(G \times X)
\end{tikzcd}
$$

where $\mathrm{Br}_e G = \ker(\mathrm{Br}_1 G \xrightarrow{e_G^*} \mathrm{Br}\, k)$

given k-gp

e.g. "Conv" (L.-Wu) $\mathcal{X} = \bigcup_{i \geq 1}^{n} [X_i / G_i]$

sm k-var

Then $\mathrm{Br}\, \mathcal{X}$ is torsion.

4.9 Rmk. For reg. Noe. DM stack $\mathcal{X}$, $\mathrm{Br}\, \mathcal{X}$ is also torsion. (Antieau – Meier '20)

4.9 "Thm" (L. - Wu) $f : Y \xrightarrow{G} X$ conn. linear lg. torsor   for var

Y : sm. geo. int

Then $X(\mathbb{A}_k)^{Br} = \bigcup_{\sigma \in H^1(k,G)} f^{\sigma}\left(Y^{\sigma}(\mathbb{A}_k)^{Br_{G^{\sigma}}(Y^{\sigma})}\right)$

Q.10

- Descent along a torsor for BM set ✓
- Product preservation ?
  - Proj sm var.   Skorobogatov - Zarhin 14
  - sm geo. int var.   L. 20
  - algebraic stacks ?

4.11 Thm (L.-Wu) The functor $-(\mathbb{A}_k)^{Br}: \mathrm{Chp}_1/k \to \mathrm{Set}$ preserves fin. prod, where $\mathrm{Chp}_1/k \subset \mathrm{Chp}/k$ full subcat spanned by sm. alg. k-stacks of f.t.

- admitting sep. geo. int atlas X s.t. $X((\mathbb{A}_k)^{Br_1}) \neq \emptyset$
- DM or Zar.-loc. quo of k-var by linear grp.

Key ingredient of proof:

- Torsionness of Br $\rightsquigarrow$ can use $H^*(-, \mu_n)$
- Existence of univ torsor of n-torsion

$$X(\mathbb{A}_k)^{Br_1} \neq \emptyset \Rightarrow \mathcal{X}: H_{fppf}(\mathcal{X}, S) \twoheadrightarrow \mathrm{Hom}_{D(k)}(\hat{S}, KD'(\mathcal{X}))$$

- Künneth formula for $H^i(-, \mu_n)$, $i = 1, 2$ $/\bar{k}$

- Künneth for stack: $Rp_* K \boxtimes^L_\Lambda Rq_* L \xrightarrow{\sim} R(p \times q)_* (K \boxtimes^L_\Lambda L)$ coh. desc

- Sm bc $p^* Rf_* \xrightarrow{\sim} Rg_* q^*$

4.12 "Cor" For stack quo of Liu–Zheng '17 sm gen. int. k-var by conn linear k-gp. Then $\mathcal{X}(\mathbb{A}_G)^{Br} \times \mathcal{Y}(\mathbb{A}_k)^{Br} \cong (\mathcal{X} \times \mathcal{Y})(\mathbb{A}_k)^{Br}$

- Descent along a torsor $\mathcal{X}(\mathbb{A}_k)^{Br} = \bigcup X(\mathbb{A}_k)^{Br_G}$
- $Br_1 \subseteq Br_G$