Thank ~~Hu~~ for inv & intro

obstructions on algebraic stacks

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§ Basic def & classical results

. When HP fails, people construct some subsets $X(k) \subseteq X(\mathbb{A}_k)^{obs} \subseteq X(\mathbb{A}_k)$, called obstructions.

~~. Here X/k var~~

. Most obs comes from a functor,

$F : (Sch_{/k})^{op} \to Set \in P(Sch_{/k})$

$q : \operatorname{Spec} \mathbb{A}_k \to \operatorname{Spec} k \in Sch_k$

we define

$$\begin{tikzcd} X(k) \arrow[rr] \arrow[d] & & X(\mathbb{A}_k) \arrow[d, "F"] \\ \boxed{} \arrow[urr, "q^*"] \arrow[d] & & \\ Map(F(X), F(k)) \arrow[r] & Map(F(X), F(\mathbb{A}_k)) & \end{tikzcd}$$

$X(\mathbb{A}_k)^F = \operatorname{im} q^*$

These defs coincide, but for in a func way, for comp on stacks

Replace $F(X)$ by some $W \subseteq F(X)$

$\leadsto X(\mathbb{A}_k)^{W}$ in par, take $W = \{\alpha\}$ so that

$X(k) \subseteq X(\mathbb{A}_k)^F = \bigcap_{\alpha \in F(X)} X(\mathbb{A}_k)^{\alpha}$.

. Point is : for many Y,

$X(\mathbb{A}_k)^{obs} \neq \emptyset \Rightarrow Y(k) \neq \emptyset$ (only) one

~~or further $X($~~

Eg

- Brauer - Manin obstruction (BM)

$$F = H^2_{\acute{e}t}(-, \mathbb{G}_{m,k}) \rightsquigarrow X(\mathbb{A}_k)^{Br}$$

- Descent obstruction

$$F = \check{H}^1_{fppf}(-, G) \qquad G - \text{linear } k\text{-gp}$$

$$X(\mathbb{A}_k)^{desc} := \bigcap_{G \text{ linear}} X(\mathbb{A}_k)^{\check{H}^1_{fppf}(-,G)}$$

replace linear by
- connected lin $\rightsquigarrow X(\mathbb{A}_k)^{conn}$
- PGL_n, $n \geq 1$ $\rightsquigarrow X(\mathbb{A}_k)^{PGL}$

~~Furthermore~~, for $[f: Y \to X] \in \operatorname{Tors}(X, G)/_{\sim} = \check{H}^1_{fppf}(X, G)$

$$X(\mathbb{A}_k)^f = \bigcup_{\sigma \in H^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k))$$

- $\mathcal{V} \subseteq$ ~~subset~~ of $\{$linear k-gps$\}$

$$\rightsquigarrow X(\mathbb{A}_k)^{\mathcal{V}\text{-}obs} := \bigcap_{\substack{G \in \mathcal{V} \\ Y \xrightarrow{G} X \text{ torsor}}} \bigcup_{\sigma \in H^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k)^{obs})$$

- $\mathcal{V}$ = all, obs = desc $\rightsquigarrow X(\mathbb{A}_k)^{desc, desc}$

- $\mathcal{V}$ = finite, obs = desc $\rightsquigarrow X(\mathbb{A}_k)^{fin, desc}$

- — — — — Br $\rightsquigarrow X(\mathbb{A}_k)^{\acute{e}t, Br}$

Homotopy obstruction

• Fix $Y \in Sch/_k$.

(Harpaz–Schlank '13) $|Y_{ét}/_{két}|$ – ét homotopy type

$Pro(k_{ét})$ – pro cat of (~~\infty~~, small) ét ~~topos~~ over $Spec\, k$

~~Let $F : (Sch/_k)^{op} \to Set$~~

Define

$X(\mathbb{A}_k)^h \to \pi_0 (Map_{Pro(\mathbb{A}_k)} (|^{k_{ét}}/_{két}|, |X_{ét}/_{két}|)$ "global section"

replace $Map(\ldots) \rightsquigarrow Map(\ldots)$ by $X(k)^h \to X(\mathbb{A}_k)^h$

$\rightsquigarrow X(\mathbb{A}_k)^h$

$\rightsquigarrow X(\mathbb{A}_k)^{h\mathbb{Z}}$ $\mathbb{Z}$-linearized ver

• We have various ~~these~~ obs …

But, essentially we do not have so many, (do we?)

(X – sm. gp, geo. int k – var)

field

$X(\mathbb{A}_k)^{Br} = X(\mathbb{A}_k)^{PGL} = X(\mathbb{A}_k)^{conn} = X(\mathbb{A}_k)^{Z\text{-desc}}$

$= X(\mathbb{A}_k)^{h\mathbb{Z}}$

Wei X simply conn. $\Rightarrow$ "="

$\cap$

$X(\mathbb{A}_k)^h = X(\mathbb{A}_k)^{fin, desc} = X(\mathbb{A}_k)^{desc, desc} = X(\mathbb{A}_k)^{ét, Br} = X(\mathbb{A}_k)^{desc}$

By Harari 02, Poonen 17, Skorobogatov 09, Demarche 09, Stoll 07, ...

Rosonen 10, Harpaz - Schlank 13,

Cao - Demarche - Xu 19, Cao 20

(If I missed some one's paper ---)

Q

3 - ~~Stack~~ obs on stack - motivation ~~motivate~~

- ~~Notation~~ Descent by torsors $\rightsquigarrow$ gerbes

Recall $X(\mathbb{A}_k)^f = \bigcup_{\sigma \in H^1(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k))$

$\mathrm{Tors}_{fppf}(Y, G)/\sim \; = \; \check{H}^1_{fppf}(X, G)$

Similarly, $\mathrm{Gerbe}_{et}(X, G)/\cong \; = H^2_{et}(X, G)$

(only consider G as commutitive k-gp

A gerbe $f: Y \longrightarrow X$ that is - is a stack over X loc. non empty loc. conn

(it is bound (lien) by some lien here we consider G,

$\cong$ means G- equivalence)

G is a scheme $\Rightarrow$ Y is ~~algebraic~~

$$\Rightarrow \quad X(\mathbb{A}_k)^f = \bigcup_{\sigma \in H^2(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k))$$

$$\Rightarrow \quad X(\mathbb{A}_k)^{2\text{-desc}} := \bigcap_{\substack{G \text{ comm} \\ k\text{-gp}}} X(\mathbb{A}_k)^{H^2_{et}(-,G)}$$

$$= \bigcap_{\substack{G \text{ comm} \\ k\text{-gp}}} \bigcup_{\sigma \in H^2(k,G)} f^\sigma(Y^\sigma(\mathbb{A}_k))$$

$$\rightsquigarrow \quad X(\mathbb{A}_k)^{2\text{-desc},\,\text{desc}} = \text{— — — —}$$

In general
So we want understand
obs on alg stacks

~~Anothe~~ Other motivation is
many moduli stack
HP for them $\longleftrightarrow$ relation bt ~~local~~ / ~~ded~~ ~~existence of~~
global ~~obs~~
geo. objs

— — —

(• An counterexample.)

Thm (Wu – C. '23)

$\exists$ inf. many (p, q) s.t.
the stacky curve $\mathcal{X}_{(p,q)}$

$= [\mathcal{Y}_{(p,q)} / \mu_2]$ violating HP

for int. pts.

- p, q primes, k # field
- $\mathcal{Y}_{(p,q)} := \mathrm{Proj}(\mathcal{O}_k[X,Y,Z] / Z^2 - pX^2 - qY^2)$
- acted on by μ_2: $[X:Y:Z] \mapsto [X:Y:-Z]$
- $\mathcal{X}_{(p,q)}$ has genus $\frac{1}{2}$
- Generalize

Bhargava – Poonen 22

/ $\mathbb{Q}$

• ~~Examples~~ of Notations & ~~Egs~~ Examples

– $Sch \subset Esp \subset Chp$.

schemes, alg spaces, alg stacks

– Simplest eg: BG classifying stack

$X = *$

$\rightsquigarrow [X/G]$ quotient stack (by mod out 2-isos)

On $Chp/_k$ we may ~~talk~~ about points, $I+P$ & obs ... (all fnd k, $k \supseteq$ clocking)

(E) the map $X(k) \to X(\mathbb{A}_k)$ is not necessary inj

eg. G ~~tor~~ k-gp. Then

$$\begin{tikzcd}
BG(k) \arrow[r] \arrow[d, equal] & BG(\mathbb{A}_k) \arrow[d, equal] \\
H^1(k,G) \arrow[r] & H^1_{fppf}(\mathbb{A}_k, G) \\
Ш^1(G/_k) \arrow[u] \arrow[r] & \{*\} \arrow[u]
\end{tikzcd}$$

in general $Ш^1(G/_k) \neq \{*\}$

Anyway we view $X(k)$ as its ess image

§ Br of some alg stacks

◦ quo stacks

Lemma (Dol.-Wais) $\mathcal{Y} = [X/G] \longleftrightarrow f: \mathcal{Y} \to X$ $\in \mathrm{Tors}(\mathcal{Y}, G)$

$$0 \to U(\mathcal{Y}) \xrightarrow{f^*} U(X) \to U(G)$$

$$\to \mathrm{Pic}\,\mathcal{Y} \xrightarrow{f^*} \mathrm{Pic}\,X \to \mathrm{Pic}\,G \to$$

$$\mathrm{Br}_1\,\mathcal{Y} \xrightarrow{f^*} \mathrm{Br}_1\,X \xrightarrow{p^* - p_2^*} \mathrm{Br}_1(G \times X)$$

PF. coh desc

$E_2^{p,q} = \check{H}^p(X/\mathcal{Y}, \mathcal{H}^q_{ét}(\mathcal{F})) \Rightarrow H^{p+q}_{ét}(\mathcal{Y}, \mathcal{F})$

–

$$\begin{tikzcd} & G \times X \arrow[r, "p"] \arrow[dl, "pr_1"'] \arrow[d, "pr_2"] & X \\ G & X & \end{tikzcd}$$

– X k-var, G connected k-gp $G \curvearrowright X$

ϕ

– $U(-) := \mathbb{G}_{m,k} / k^* \in \mathrm{PAb}(\mathrm{Sch}/k)$

$k^* \leftarrow$ constant presh

$\mathrm{Pic}(-) = H^1_{ét}(-, \mathbb{G}_{m,k}) \in \mathrm{PAb}(\mathrm{Sch}/k)$

– $U(G) = \hat{G}(k)$ if G is further linear

– Ex Fnel

Sansuc §?

$\rightsquigarrow$ **Cor** Let $X = \mathrm{Spec}\,k$, $\mathcal{Y} = BG$

$\big(U(BG) = 0, \ \mathrm{Pic}\,BG = U(G) \big)$

$$0 \to \mathrm{Pic}\,G \to \mathrm{Br}_1(BG) \to \mathrm{Br}\,k \to 0 \quad \text{split}$$

~~Consider BE, E ell, $\Rightarrow$ conn is crucial~~

• ~~the known~~ For a reg. sch X, ~~Noetherian~~ fin. Noetherian

$\mathrm{Br}\, X$ is torsion — Grothendieck

Cor $\mathcal{X} \in \mathrm{Chp}/k$

$$\mathcal{X} = \bigcup_{i=1}^{n} [X_i / G_i] \quad \text{open substacks}$$

- X_i — sm k-sch.
  $\circlearrowleft$
- G_i — linear k-gp ~~acting~~

$\Rightarrow$ $\mathrm{Br}\, \mathcal{X}$ is torsion.

Rmk — linear is crucial

counter-eg: BE, E ell. (s.t. $\mathrm{rk}\, E \geq 1$) $\mathrm{Br}(BE) \cong \mathrm{Br}\, k \oplus \mathrm{Pic}\, E$

$\underbrace{\mathrm{Pic}^0 E \oplus \mathbb{Z}}_{= \mathrm{Jac}\, E = E}$

- Antieau–Meier 20 $\mathcal{X}$ reg. Noe Deligne–Mumford (DM) stack

$\Rightarrow$ $\mathrm{Br}\, \mathcal{X}$ tor.

§ Obs on stacks — ~~Comp~~ Relations ~~of obs~~

Thm (L.-Wa23, Wu-L.24)

~~$\mathcal{X}$ $\mathcal{O}_k$ alg. stc~~

$\mathcal{X} \in \mathrm{Chp}_{/k}$

ext. Harari 02 ① $\mathcal{X}(\mathbb{A}_k)^{\mathrm{Br}} \subseteq \mathcal{X}(\mathbb{A}_k)^{\mathrm{cons}}$.

② $\mathcal{X}$ of f.t., ~~and~~ or ~~either~~

~~DM of f.t.~~, Zar-locally ~~fin~~

f.t. ⇒ finitely many

$\bigcup_{i=1}^{n} [Y_i / G_i]$ $\quad Y_i$ - k-var

G_i - lin. k-gp.

⇒ $\mathcal{X}(\mathbb{A}_k)^{\mathrm{Br}} = \mathcal{X}(\mathbb{A}_k)^{2\text{-desc}}$

ext Sk09 CDX19 ③ $\mathcal{X}$ of f.t., DM.

⇒ $\mathcal{X}(\mathbb{A}_k)^{\mathrm{desc}} = \mathcal{X}(\mathbb{A}_k)^{\mathrm{fin.\,desc}}$.

§ Properties of $\mathcal{X}(\mathbb{A}_k)^{Br}$

- Descent
- Cao 18 defines inv. Br_G as Br_1

- for $\mathcal{X} = [Y/G]$

Y — sm geo. int k-var
G — conn lin. k-gp

$\rightsquigarrow f : Y \xrightarrow{G} \mathcal{X} \in \mathrm{Tors}(\mathcal{X}, G)$

$\rightsquigarrow$ can also def.

$Br_G \mathcal{X} := \{ b \in Br\, \mathcal{X} \mid p^* b - p_2^* b \in Br\, G \}$

$\Rightarrow$ Modified Sansuc exact seq hold ✓

$\Rightarrow$ Ext. Cao 18. D

Thm $\mathcal{X}(\mathbb{A}_k)^{Br} = \bigcup_{\sigma \in H^1(k, G)} f^\sigma \left(Y^\sigma(\mathbb{A}_k)^{Br_{G^\sigma}(Y^\sigma)} \right)$

(fin)

∘ Products (~~fin many~~)

X, Y sm. geom. int. k-vars.

Q

$$(X \times_k Y)(\mathbb{A}_k)^{Br} \overset{?}{=} X(\mathbb{A}_k)^{Br} \times Y(\mathbb{A}_k)^{Br}$$

— X, Y projective — Skorobogatov–Zarhin 14

— any X, Y — L. 20

Thm (L.–Wu 23) Q

$\mathcal{X}$, $\mathcal{Y}$ are stacky quot. of sm. geo. int. k-vars by conn. lin. k-gps.

(ie, $\mathcal{X} = [X/G_X]$ & $\mathcal{Y} = [Y/G_Y]$, G_X, G_Y)

Pf ① $H^1_{fppf}(\mathcal{X}, S) \xrightarrow{\sim} \mathrm{Hom}_{D(k)}(\hat{S}, KD'(X))$ is surj.

Q

(2) ~~Kun~~ Künneth for stacks.

(3) $*$-BM) BC for stacks

(4) descent for BM. sett

Open problems

For $X \in \mathrm{Chp}/k$.

~~when we have~~

(1) $X(\mathbb{A}_k)^{\text{ét}, \mathrm{Br}} = X(\mathbb{A}_k)^{\text{desc}} = X(\mathbb{A}_k)^{\text{desc, desc}}$

(2) $X(\mathbb{A}_k)^{2\text{-desc}, \text{desc}} \subsetneq X(\mathbb{A}_k)^{\text{desc}}$? for some X ?

Time permitting part

§+ Again relations + coh des approach

⚡ ~~in progress, May have mistakes!~~ just idea. discuss

series arXiv

- Liu - Zheng (12-24)

~~six functor fo~~

six operations ~~for~~

(n-)(∞A)-cat (higher) ~~Artin~~ stacks

six-functor formalisms

Heyer - Mann 24 (collection)

Gulotta - Hansen - Weinstein 22 (v-stacks)

Scholze 22/23

Gaitsgory - Rozenblyum 19

an (∞,2)-cat

① $D : N(\mathrm{Chp})^{op} \longrightarrow \mathrm{Pr}^{L}_{st}$

$X \longmapsto D(X,\Lambda)$ s.t. $h(D(X,\Lambda)) = D_{cart}(X_{lis\acute{e}t},\Lambda)$

$f : Y \to X \rightsquigarrow f^*$

② f^* f_*

$f_!$ $f^!$

$\otimes$ $\mathcal{H}om$

($\Lambda \in$ Coeff. Ring ⊠-tor ⊠-Coprim.)

③ Künneth, $\otimes_!$-bc $\hookrightarrow$ proper

(sm)$_*$-bc   proj formula

Poincaré duality

$f^*\langle \dim f\rangle \cong f^!$

④ <u>coh descent</u> (④ dates back to Deligne SGA4 V bis)

$f: X_0 \to X_{-1}$ sm surj

$\rightsquigarrow$ $X_\bullet : N(\Delta_+)^{op} \to N(\mathrm{Chp})$ Čech nerve

$\Rightarrow$ $D(X_{-1}, \Lambda) \xrightarrow{\sim} \varprojlim_{n\in\Delta} D(X_n, \Lambda)$.

$*$-pullback.

- Consider $\mathrm{Chp}_{/S}$ where $g: A \to S$ a (think of $\mathrm{Spec}\, A_k \to \mathrm{Spec}\, k$)

$X_1 \rightrightarrows X_0 \xrightarrow[\text{sm surj}]{f} X_{-1}$ Čech,

use extend from Sch $\Rightarrow$ Esp $\Rightarrow$ Chp

$\rightsquigarrow$ $D(X_{-1}) \xrightarrow{f^*} D(X_0) \rightrightarrows D(X_1) \cdots$ lim diag

$D(X_{-1}, \Lambda)$

$\Rightarrow$ $\mathcal{H}(X_{-1}) \xrightarrow{f^*} \mathcal{H}(X_0) \rightrightarrows \mathcal{H}(X_1) \cdots$ $\in \mathrm{Pr}^L$

$\mathcal{H}(X_{-1}) = \mathrm{Fun}(D(A, \Lambda), D(X_{-1}, \Lambda))$

$\mathcal{H}: N(\mathrm{Chp}_{/S})^{op} \to \mathrm{Pr}^{L\otimes}$

For $f : Y \to X \in \mathrm{Chp}/S$? (Coz. f.t. Π-for)

$$\mathcal{H}(X) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{H}(Y)$$

$$\mathcal{H}(Y) \underset{f^!}{\overset{f_!}{\rightleftarrows}} \mathcal{H}(X)$$

Want. $\mathcal{A}(X)^{obs} \subset \mathcal{H}(X)$ $:= \{ x_* \in \mathcal{H}(X) \mid x \in X(A)^{obs} \}$

But (even make it presentable) and for + colim. $X(A)$)

it can not descent!

$$\mathcal{H}(X_{-1}) \to \mathcal{H}(X_0) \rightrightarrows \mathcal{H}(X_1) \cdots$$

$$0$$

$$X_{-1}(A) \xleftarrow{} X_0(A) \rightleftarrows X_1(A) \cdots$$

can not ob

$$0 \qquad 0 \qquad 0 \cdots$$

Resolve trick

1. $\mathcal{B}(X)^{obs}$:= spanned by x_*, s.t. $\exists\, x_0 \in X(A)^{obs}$ & $x_* \to x_{0*}$.

∩ strictly full sub

$\mathcal{H}(X)$

- for ϕ:

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ A \arrow[ur, "y_0"] \arrow[r, "x_0"'] & X \end{tikzcd}$$

if x_0 lifts to some $y_0 \in Y(A)^{obs}$

$$f^* x_{0*} = f^* f_* y_{0*} \xrightarrow[\text{counit}]{} y_{0*}$$

$\rightsquigarrow$ we have pb

$$f_x : \mathcal{B}(X)^{obs} \to \mathcal{B}(Y)^{obs}$$

– unit $\Rightarrow$ $x_* \in \mathcal{U}(X)$, $f^* x_* \in \mathcal{B}^{obs}(Y)$ ...

$$\Updownarrow$$

$$x_* \in \mathcal{B}^{obs}(X)$$

~~the property $Y(A)^{obs} \to X(A)^{obs}$ is stable under~~

<u>But</u> in general we can not lift every $x \in X(Y)^{obs}$ (even for $X(\mathbb{C})$, $X(A)$)

② we may "construct" sheafification of

$$\left(X \in Sch \longmapsto A^{obs}(X) \right)$$

formally.. let

$$\mathcal{C}^{obs} = \varinjlim_{K} F \in Fun^{\mathcal{E}}(N(Sch_S)^{op}, Cat_\infty)$$

descent for edge in $\mathcal{E}$

K = spaned by F, $A^{obs} \subseteq F \subseteq \mathcal{H}^{\otimes}$

of full $Fun^{\mathcal{E}}(N(Sch_S)^{op}, Cat_\infty^{\otimes})$

Then use $Fun^{\mathcal{E}}(N(Sch_S)^{op}, Cat_\infty) \xrightarrow{\sim}$

$Fun(\ldots Esp_S) \xrightarrow{\sim}$

$-(-\ Chp_S)$

to extends to

$\widetilde{\mathcal{E}}^{obs} : N(\mathrm{Chp}_{/S})^{op} \longrightarrow \mathrm{Cat}_\infty$
$\mathcal{H}$

s.t. ① ② descent for $\widetilde{\mathcal{E}}$

③ $f : Y \to X \rightsquigarrow f^*$

$\Rightarrow$ $\widetilde{\mathcal{E}}^{obs}$ preserves ves from $\mathcal{H}$

inclusion $\forall X \in \mathrm{Sch}$

$X(A)^{obs_1} \subseteq X(A)^{obs_2}$

Q How to include $X(S)$

making $X(A)^{\widetilde{obs}} :=$

$\{ x \in X(A) \mid x_* \in \widetilde{\mathcal{E}}^{obs}(X) \}$

an obstruction on Chp?

— At least we can lift S-pts.

But ~~that~~ not stable under forming diagonals ... (this cond st. under comp & fc)

③ use semisimplicial descent ...

$N(\Delta_s) \subset N(\Delta)$ is cofinal

$\Rightarrow$ $\mathcal{H}(X_{-1}) \longrightarrow \mathcal{H}(X_0) \rightrightarrows \mathcal{H}(X_1) \cdots$

also limit

ie $\mathcal{H}(X_{-1}) \xrightarrow{\sim} \lim_{n \in \Delta_s} \mathcal{H}(X_n)$

~~spine in Δ_S ... (X) ... com pt~~

<u>Upshot</u> $\exists$ functor

$$e^{obs} : N(\mathrm{Chp}^{\mathcal{E}, \mathrm{l}\#,\square}_{/S})^{op}_{\mathcal{E}} \to \mathrm{Cat}_\infty .$$

$\mathcal{H}(-,\Lambda)^{\text{S.f.}}$, $\forall f : Y_0 \to X_{-1}$ in $\mathcal{E}$,

and $X_\bullet\, N(\Delta_{s,+})^{op} \to N(\mathrm{Chp}^{\mathrm{l}\#,\square}_{/S})_{\mathcal{E}}$.

semi simplicial Čech nerve,

$$e^{obs}(X_{-1}) \xrightarrow{\sim} \varprojlim_{n \in \Delta_s} e^{obs}(Y_n)$$

via $*$-pb.

— $\mathcal{E} \subseteq \mathcal{F} \subseteq N(\mathrm{Chp}_{/S})_1$ stable under comp & bc.
(deg. edges $\subseteq$?

$\forall f \in \mathcal{F}$ $: Y \to X$, f rep.

$\forall f \in \mathcal{E}$, f sm surj &

Nisnevich surj $\mathcal{Y}(S) \twoheadrightarrow X(S)$

— $\Lambda \in \mathrm{Ring}_{\square\text{-tor}}$, $\mathrm{Chp}^{\mathcal{E},\mathrm{l}\#,\square}_{/S}$ spanned by

$\square$-equiv S-stacks loc. f.t. adm. $\mathcal{E}$-atlases

— ~~$X(A)$~~ $e^{obs}(X) \simeq A^{obs}(X)$
for $X \in \mathrm{Sch}^{\mathrm{l}\#,\square}_{/S}$

— $X(A)^{\widetilde{obs}} := \{ x \in X(A) \mid x_* \in e^{obs}(X) \}$
$\forall Y$
$X(S)$, $\forall X$

— e^{obs}, $X(A)^{\widetilde{obs}}$ preserves
② "the only one" ① relation of obs on Sch

(4) How to show $\tilde{X}(A)^{obs} \subseteq X(A)^{obs}$

– restrict to

$$\begin{tikzcd} & & & P_{Y_*}x'_* \arrow[r] \arrow[d] & q_* \\ x'_* \to x_* \in \Sigma t & s.t. & q_* \arrow[ur] \arrow[dr] \arrow[urr, bend left, "id"] & P_{Y_*}x_* \arrow[ur] & \end{tikzcd}$$

$\rightsquigarrow$ for $obs = H^i_{et}(-, G)$

$\uparrow$ comm. fin-tor

indeed $*$-pb is same
$$H^i_{et}(X, G) \underset{\cong}{\overset{x^*}{\rightleftarrows}} H^i_{et}(S, G)$$

Suff to show (restricted) mod:

$\Rightarrow$ ~~Want~~ $\checkmark$ B^{obs} descent so that

$\mathcal{C}^{obs} \hookrightarrow B^{obs}$?

How to ?